Shaping optical beams with non-integer orbital-angular momentum: a generalized differential operator approach

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We introduce an analytical procedure to construct an optical beam with an arbitrary value of orbital-angular momentum (OAM) by keeping the flexibility of shaping its transverse intensity distribution without changing its OAM. We apply the general theory of fractional differential operators in Fourier domain to derive general expressions for the OAM content in the beam and find the relevant parameters that determine its OAM value and those that can be freely modified without affecting it.

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Orbital-angular momentum (OAM) is a fundamental property of a light beam, and its characteristics and applications have been studied extensively since the seminal paper by Allen *et al.* in 1992 [1,2]. In the paraxial domain, the z component of OAM per photon in unit length about the origin of a transverse slice of a scalar beam $U(\mathbf{r}, z)$ is given by [2]

$$J_z = \frac{\iint_{-\infty}^{\infty} \mathbf{r} \times \mathrm{Im}(U^* \nabla_\perp U) \mathrm{d}x\mathrm{d}y}{\iint_{-\infty}^{\infty} |U|^2 \mathrm{d}x\mathrm{d}y}, \tag{1}$$

where $\mathbf{r} = (x, y) = (r \cos\theta, r \sin\theta)$ is the transverse radius vector, $\nabla_\perp = (\partial_x, \partial_y)$ is the transverse gradient, and ∂_x is the partial derivative with respect to x. While individual light modes with azimuthal dependence $\exp(\mathrm{i}m\theta)$, where $m = 0, \pm 1, \pm 2, \ldots$, carry integer OAM per photon (in units of $\hbar$), it is well known that a generic optical beam may transport arbitrary non-integer, or fractional, OAM [3–10].

Optical beams with non-integer values of OAM have been generated in several ways. For example, using non-integer spiral phase plates [3,4], suitable superpositions of light modes $U_m(r)\exp(\mathrm{i}m\theta)$ with different values of m [5–7], differential operators acting on Gaussian and Bessel beams [8–10], astigmatic elements [11,12], and finite superpositions of Hermite–Gaussian and Laguerre–Gaussian beams [13,14]. All these procedures are successful in generating a beam with a given value of OAM, but the method presented here offers a much greater flexibility to modify the beam shape keeping the OAM constant. The method has potential applications in optical trapping, micromanipulation, and transfer of OAM by structured light fields.

In this Letter, we address the problem of shaping an optical beam given an arbitrary value of its OAM. Our objective is to develop an analytical procedure to generate an optical beam with a particular value of OAM by keeping the flexibility of shaping its transverse intensity distribution without changing its OAM. To this end, we apply the theory of differential operators in Fourier domain to define a creation operator of the beam whose radial and angular parts can be adjusted to shape the intensity pattern. The results reported in this Letter consolidate and generalize previous analysis on optical beams with non-integer OAM [3–14].

We begin the analysis by considering an axially symmetric seed beam $U_0 = U_0(r, z)$ that satisfy the paraxial wave equation (PWE)

$$(\partial_x^2 + \partial_y^2 + \mathrm{i}2k\partial_z)U_0 = 0, \tag{2}$$

where k is the wave number. The fundamental Gaussian beam and the zeroth-order Bessel–Gauss beam are typical examples of seed functions $U_0(r, z)$. From the purely radial dependence of $U_0(r, z)$, it is clear that the seed beam does not carry OAM.

Now, let $\hat{\mathcal{A}}(\mathcal{D}_x, \mathcal{D}_y)$ be a linear differential operator that can be expressed in general as a sum of products of derivatives of the form

$$g(a, b)\mathcal{D}_x^a \mathcal{D}_y^b, \tag{3}$$

where $\mathcal{D}_x^a$ and $\mathcal{D}_y^b$ denote the ath-order and the bth-order derivatives with respect to x and y, respectively, the parameters a and b are real numbers, and $g(a, b)$ is an amplitude factor. For generality, the order of the derivatives not necessarily has to be integer, but it may be fractional [15]. Formally speaking, we adopt the short notation $\mathcal{D}_x^a$ to represent the differential Riemann–Liouville operator $_{-\infty}\mathcal{D}_x^a$ with lower bound at $-\infty$ [15]. Remember that the fractional derivative $\mathcal{D}_x^a$ becomes the ordinary integer-order derivative when a is an integer.

The possibility of choosing several terms of the form (3) with different values for the orders of the derivatives allows to get many different realizations of the operator $\hat{\mathcal{A}}(\mathcal{D}_x, \mathcal{D}_y)$. For example, the known ladder operators

$$\hat{\mathcal{L}}^\pm = \partial_x \pm \mathrm{i}\partial_y, \tag{4}$$

are special cases of $\hat{\mathcal{A}}$.

The operator $\hat{\mathcal{A}}$ commutates with the operator of the PWE, therefore, the action of $\hat{\mathcal{A}}$ on the seed function $U_0(r, z)$ will give a new solution

$$U(\mathbf{r},z) = \hat{A}(\mathcal{D}_x,\mathcal{D}_y)U_0(r,z), \tag{5}$$

of the PWE. In this way, $\hat{A}$ plays the role of a *creation operator* of the wave function U starting from the seed function U_0. For example, the derivatives ∂_x and ∂_y generate the elegant Hermite–Gaussian beams from the Gaussian beam, and the ladder operators $\hat{\mathcal{L}}^{\pm}$ generate the elegant Laguerre–Gauss and Bessel beams from the Gaussian and the J_0 Bessel beams, respectively [8,9].

Our first goal is to find an expression of the OAM carried by the field $U = \hat{A}U_0$ in terms of the creation operator $\hat{A}$. This expression should be obtained by substituting Eq. (5) into Eq. (1). However, although this procedure is technically correct, it appears that the explicit evaluation of the double integral for an arbitrary differential operator $\hat{A}$ cannot be performed in closed form. Instead of dealing directly with the evaluation of Eq. (1), we take advantage of the properties of the linear differential operators in Fourier space to get information of the OAM carried by the beam $U(\mathbf{r},z)$.

Let $\mathbf{k} = (k_x,k_y) = (\rho\cos\phi,\rho\sin\phi)$ be the Cartesian and polar coordinates in Fourier space. If $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ denote the two-dimensional Fourier transform and its inverse, then the transformation $U = \hat{A}U_0$ can be evaluated in Fourier domain with the prescription

$$U = \mathfrak{F}^{-1}\{\mathfrak{F}\{\hat{A}U_0\}\} = \mathfrak{F}^{-1}\{\mathcal{A}\mathfrak{F}\{U_0\}\} = \mathfrak{F}^{-1}\{\mathcal{A}\tilde{U_0}\}, \tag{6}$$

where $\tilde{U}(\mathbf{k};z)$ and $\tilde{U_0}(\rho;z)$ are the Fourier transforms of the beam $U(\mathbf{r},z)$ and the seed beam $U_0(r,z)$, respectively, and the algebraic function $\mathcal{A}(k_x,k_y) = \mathcal{A}(\rho,\phi)$ is the representation of the creation operator $\hat{A}(\mathcal{D}_x,\mathcal{D}_y)$ on the Fourier plane (k_x,k_y) obtained with the substitution [15]

$$\mathcal{D}_x^a \leftrightarrow \left(\begin{array}{ll} (\mathrm{i}k_x)^a = (\mathrm{i}\rho\cos\phi)^a, & a \ge 1, \\ (-\mathrm{i}k_x)^a = (-\mathrm{i}\rho\cos\phi)^a, & a < 1. \end{array} \right. \tag{7}$$

By replacing Eq. (6) into Eq. (1), we have demonstrated that the OAM carried by U can be written as

$$J_z = \frac{\iint |\tilde{U_0}|^2 \mathcal{A}^* \partial_\phi \mathcal{A} \mathrm{d}\mathbf{k}}{\mathrm{i}\iint |\tilde{U_0}|^2 |\mathcal{A}|^2 \mathrm{d}\mathbf{k}}, \tag{8}$$

where $\mathrm{d}\mathbf{k} = \mathrm{d}k_x\mathrm{d}k_y = \rho\mathrm{d}\rho\mathrm{d}\phi$ is the differential element of area, and the integrals are carried out over the whole transverse Fourier plane.

Equation (8) is the first important result of this work. It gives the OAM carried by the beam $U(\mathbf{r},z)$ in terms of the angular spectrum of the seed function $U_0(r)$ and the algebraic factor $\mathcal{A}(\rho,\phi)$ in Fourier space. As far as we know, the expression (8) has not been reported in the optics literature. An important advantage of this representation is that a beam with a particular value of OAM may be shaped by selecting the adequate operator $\mathcal{A}$, and for most cases this is easier in $\mathbf{k}$ space than in configuration space $\mathbf{r}$.

An important special case of Eq. (8) occurs when the algebraic operator $\mathcal{A}(\rho,\phi)$ is separable in polar coordinates, that is

$$\mathcal{A}(\rho,\phi) = \mathcal{R}(\rho)\Phi(\phi). \tag{9}$$

For example, the Fourier representation of the operators $\hat{\mathcal{L}}^{\pm}$ is given by $\mathcal{L}^{\pm} = (\mathrm{i}k_x \mp k_y) = \mp\mathrm{i}\rho\exp(\pm\mathrm{i}\phi)$, so it belongs to the class of separable operators in $\mathbf{k}$ space.

Substitution of Eq. (9) into Eq. (8) leads to the cancellation of the radial integrals, then J_z reduces to

$$J_z = \frac{-\mathrm{i}\int \Phi^* \partial_\phi \Phi \mathrm{d}\phi}{\int |\Phi|^2 \mathrm{d}\phi}, \tag{10}$$

where the integrals are carried out over the whole domain of the angular variable ϕ. We conclude that, for separable operators $\mathcal{A} = \mathcal{R}\Phi$, the OAM depends exclusively on the angular part $\Phi(\phi)$ of the operator. Thus we can adjust freely the radial part $\mathcal{R}(\rho)$ or the seed function U_0 to shape the transverse pattern of the beam U by keeping the OAM constant. Equation (10) resembles the quantum-mechanical expression for the expectation value of angular momentum of a wave function $\Phi(\phi)$, but note that in our context, the OAM operator $-\mathrm{i}\partial_\phi$ is evaluated in Fourier domain and it is valid only when the operator $\mathcal{A}$ is separable.

The OAM J_z in Eq. (10) must be a real quantity for any arbitrary function $\Phi(\phi)$. To verify this, let us express $\Phi(\phi)$ in terms of its magnitude $F(\phi) \equiv |\Phi|$ and phase $\Omega(\phi) \equiv \arg\Phi$, i.e.,

$$\Phi = F\exp(\mathrm{i}\Omega). \tag{11}$$

Because in general Φ is a multivalued function, we have to define a branch cut. Although it can be located at any angular position, for convenience, we locate it along the line $\phi = \pi, -\pi$, i.e., at the negative k_x axis. Without loss of generality, we will assume that F and Ω are continuous functions of ϕ in the interval $(-\pi,\pi]$, and that the only possible discontinuity of these functions may occur just at the branch cut. Below we will discuss the case of having several discontinuities within the interval.

Replacing Eq. (11) into Eq. (10) gives

$$\int \Phi^* \partial_\phi \Phi \mathrm{d}\phi = H + \int_{-\pi}^{\pi} F\partial_\phi F \mathrm{d}\phi + \mathrm{i}\int_{-\pi}^{\pi} F^2 \partial_\phi \Omega \mathrm{d}\phi, \tag{12}$$

where H is the contribution to the integral of the discontinuity at $\phi = \pi, -\pi$. The first integral in Eq. (12) can be evaluated by applying integration by parts, we get

$$\int_{-\pi}^{\pi} F\partial_\phi F \mathrm{d}\phi = [F(\pi)^2 - F(-\pi)^2]/2. \tag{13}$$

To calculate H, observe that at the discontinuity line, we have the derivative of a step function, thus $\partial_\phi\Phi = [\Phi(-\pi) - \Phi(\pi)]\delta(\phi-\pi)$, where δ is the Dirac Delta function. Φ^* is also discontinuous at the line $\phi = \pi, -\pi$, but in this case, we just need to take the average of Φ^* at the discontinuity [16], i.e., $\Phi^* = [\Phi^*(-\pi) + \Phi^*(\pi)]/2$. In this way, the contribution to the integral of the discontinuity is the product $[\Phi(-\pi) - \Phi(\pi)][\Phi^*(-\pi) + \Phi^*(\pi)]/2$ that simplifies to

$$H = \frac{1}{2}[F(-\pi)^2 - F(\pi)^2] + \mathrm{i}F(\pi)F(-\pi)\sin(\Delta\Omega), \tag{14}$$

where $\Delta\Omega \equiv \Omega(-\pi) - \Omega(\pi)$ is the phase jump at the discontinuity line.

Coming back to Eq. (12), we see that the evaluation of the integral (13) cancels out with the first term of H, therefore Eq. (10) becomes

$$J_z = \frac{\int_{-\pi}^{\pi} F^2 \partial_\phi \Omega \mathrm{d}\phi + F(\pi)F(-\pi)\sin(\Delta\Omega)}{\int_{-\pi}^{\pi} F^2 \mathrm{d}\phi}, \tag{15}$$

which clearly is a real number. We remark that this procedure can be applied to Eq. (8) to prove that it yields a real number as well.

Equations (10) and (15) offer two different but fully equivalent ways of calculating the same OAM. Whereas the former is more appropriate for continuous functions $\Phi(\phi)$, the latter is more *useful* when the angular function $\Phi(\phi)$ needs a branch cut, i.e., it tells us the explicit OAM contribution of the phase jump $\Delta\Omega$. Note that this contribution vanishes when $\Delta\Omega$ is a multiple of π, which produces a single valued function without the branch cut.

Equation (15) can be further simplified if we assume that the amplitude $F(\phi)$ of the angular function $\Phi(\phi)$ is constant. Again, an example of this situation is the ladder operator $\mathcal{L}^{\pm} = \mp \mathrm{i}\rho \exp(\pm \mathrm{i}\phi)$ for which $F(\phi) = 1$. When $F(\phi) = \text{cons}$, the integrals in Eq. (15) are easily evaluated and the OAM reduces to

$$J_z = [\sin(\Delta\Omega) - \Delta\Omega]/2\pi. \tag{16}$$

It is worthwhile to mention that Eq. (16) was reported previously for optical beams with fractional OAM generated with spiral phase masks [3,4], superposition of OAM modes [6,7], and ladder operators $\mathcal{L}$ [8,9], but we remark that all these realizations are actually special cases of the general formalism introduced in this work. In fact, Eq. (16) gives the OAM for all beams generated with algebraic operators of the form $\mathcal{A}(\rho,\phi) = \mathcal{R}(\rho)\exp[\mathrm{i}\Omega(\phi)]$, and the value depends only on the phase jump $\Delta\Omega$ at the discontinuity line. In other words, once fixed the value of $\Delta\Omega$, the behavior of $\Omega(\phi)$ and $\mathcal{R}(\rho)$ can be modified to shape the beam without changing the value of the OAM. This situation is illustrated in Fig. 1(a) for several curves $\Omega(\phi)$. From Eq. (16), it is clear that J_z will be integer only if $\Delta\Omega$ is a multiple of 2π.

The function $\Omega(\phi)$ is arbitrary and is not constrained to be continuous everywhere on its domain $(-\pi, \pi]$. As shown in Fig. 1(b), if $\Omega(\phi)$ has N discontinuities located at positions ϕ_n, then by taking into account all phase jumps, it can be proved that Eq. (16) generalizes to

$$J_z = \sum_{n=1}^{N}[\sin(\Delta\Omega_n) - \Delta\Omega_n]/2\pi, \tag{17}$$

where $\Delta\Omega_n \equiv \Omega(\phi_n^+) - \Omega(\phi_n^-)$ is the nth phase jump.

The consideration of discontinuous functions adds flexibility to the beam shaping process. For example, suppose we want to shape a beam with a given OAM

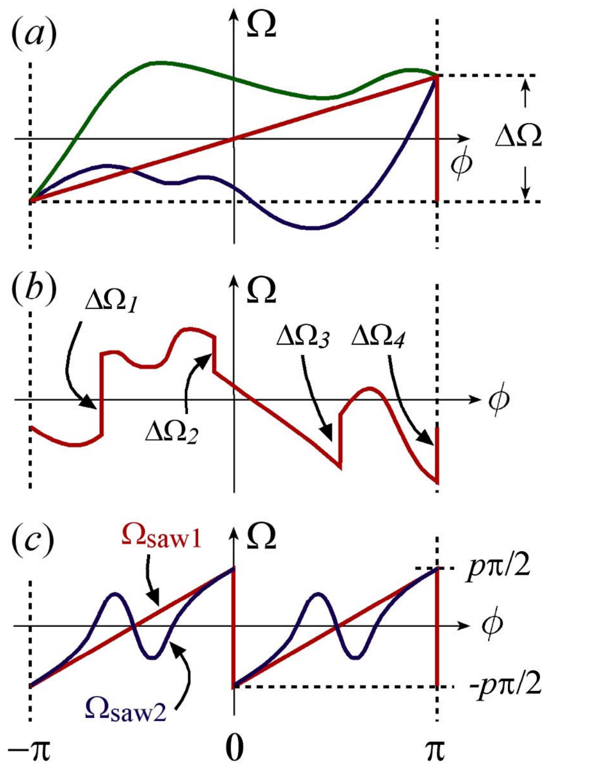

Fig. 1. (a) Different curves $\Omega(\phi)$ with the same phase jump $\Delta\Omega$ generate beams with the same OAM but different transverse shape. (b) Phase function $\Omega(\phi)$ with several discontinuities within its domain. (c) Sawtooth functions $\Omega_{\mathrm{saw1}}(\phi) = p[(\phi+\pi)\mathrm{mod}\pi] - p\pi/2$, and $\Omega_{\mathrm{saw2}}(\phi) = \Omega_{\mathrm{saw1}} + 0.5\mathrm{se}_2(\phi, 30)$, where se_2 is the odd Mathieu function of order 2.

and whose transverse intensity distribution has two perpendicular axes of symmetry, that is $|U(r,\theta,z)|^2$ remains invariant under the transformations $\theta = \beta \to -\beta$ and $\theta = \frac{\pi}{2} + \beta \to \frac{\pi}{2} - \beta$. We will show a way to achieve this, starting from the field U expressed in terms of the operator $\mathcal{A}$ and the seed function U_0 through an inverse Fourier transform Eq. (6)

$$U(r,\theta,z) = \iint \mathcal{A}(\rho,\phi)\tilde{U_0}(\rho;z)e^{\mathrm{i}\rho r\cos(\phi-\theta)}\mathrm{d}\mathbf{k}. \tag{18}$$

For separable $\mathcal{A}(\rho,\phi) = R(\rho)\Phi(\phi) = R(\rho)\exp[\mathrm{i}\Omega(\phi)]$, the only concern is the integral in ϕ because it is the only place where θ appears, therefore the absolute value of $\int_{-\pi}^{\pi}\Phi(\phi)\exp[\mathrm{i}\rho r\cos(\phi-\theta)]\mathrm{d}\phi$, should remain invariant under the required transformations. Invoking angle sum identities, we identify that this condition is satisfied by antisymmetric functions $\Omega(\phi)$ about $\phi = -\pi/2, 0, \pi/2, \pm\pi$, as for example the functions shown in Fig. 1(c), where Ω_{saw1} and Ω_{saw2} are sawtooth functions with constant and non-constant ramps, respectively. The sawtooth functions have two discontinuities with $\Delta\Omega = -p\pi$, thus from Eq. (17) the OAM for the two-axes reflective beam is given by

$$J_z = p - \sin(p\pi)/\pi, \tag{19}$$

where the value of p is determined once the desired value of J_z is given. We then conclude that all beams generated

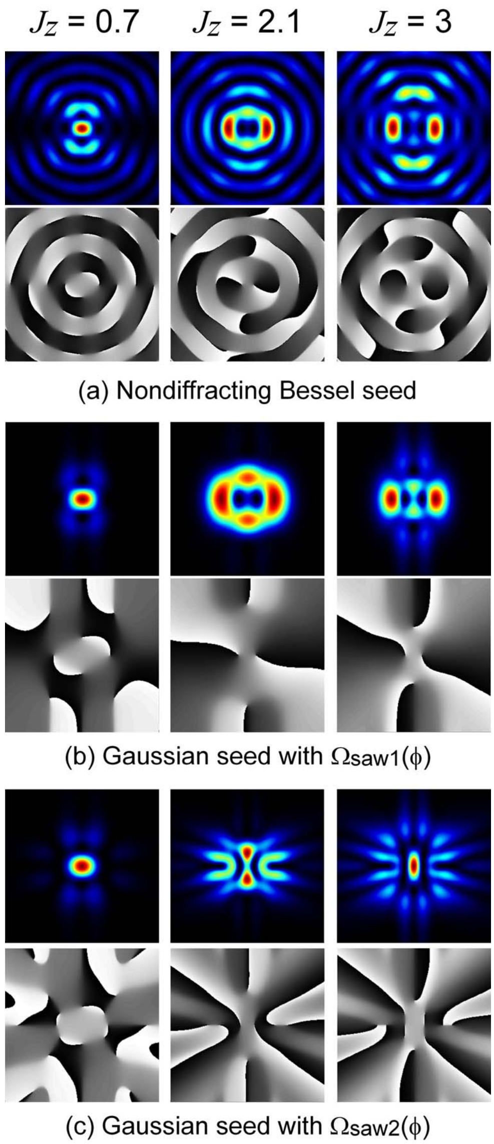

Fig. 2. Intensity and phase at $z = 0$ for reflective symmetric beams with $J_z = 0.7$, 2.1 and 3.0. (a) Nondiffracting type generated with a J_0 Bessel beam as seed beam. (b) and (c) Gaussian-type generated with a Gaussian beam as seed function, $R(\rho) = \rho^2$ as radial variation, and $\Omega_{\mathrm{saw1}}(\phi)$ and $\Omega_{\mathrm{saw2}}(\phi)$ as angular variations, respectively.

with the operator $\mathcal{A}(\rho, \phi) = R(\rho)\exp[\mathrm{i}\Omega_{\mathrm{saw}}(\phi)]$ in Eq. (18) will have axes of symmetry along the x and y axes and OAM J_z. Note that the radial function $R(\rho)$, the seed beam U_0, and the behavior of the ramps of $\Omega_{\mathrm{saw}}(\phi)$ are free to be modified without changing the OAM or the reflective property. These features are illustrated in Fig. 2 where we show the intensity and phase patterns of beams with $J_z = 0.7$, 2.1, and 3.0. In subplot (a), we display a nondiffracting beam generated with the zeroth-order Bessel beam as seed beam $[\tilde{U_0}(\rho) = \delta(\rho - \rho_0)/\rho]$, $R(\rho) = 1$ as radial variation, and $\Omega_{\mathrm{saw1}}(\phi)$. In subplots (b) and (c), we show Gaussian-type beams generated with the Gaussian beam as seed beam $[\tilde{U_0}(\rho) = \exp(-\rho^2/\rho_0^2)]$, $R(\rho) = \rho^2$ as radial function, and angular variations $\Omega_{\mathrm{saw1}}(\phi)$ and $\Omega_{\mathrm{saw2}}(\phi)$, respectively.

In conclusion, we introduced an analytical procedure to shape an optical beam with a particular value of OAM by keeping the flexibility of shaping its transverse intensity distribution without changing its OAM. In the process of deriving the method, we determined general expressions for the OAM content carried by the light beams in terms of the creation operator of the beam and the seed function. From these expressions, we proved that the phase discontinuities of the angular part of the creation operators play a crucial role in the determination of the OAM.

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