Orbital angular momentum of optical vortices from power measurements and the cross-correlation function

Benjamin Perez-Garcia,[1] Dorilian Lopez-Mago,[2] Hipolito Garcia-Gracia,[1,*] Jorge A. Garza-Alanis,[1] Raul I. Hernandez-Aranda,[1] and Julio C. Gutiérrez-Vega[1]

[1]*Photonics and Mathematical Optics Group, Tecnológico de Monterrey, Monterrey 64849, Mexico*

[2]*ETH Zürich, Photonics Laboratory, 8093 Zürich, Switzerland*

**Corresponding author: hgarciag@itesm.mx*

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We show that the complex-amplitude cross-correlation function between two beams can be obtained by the global Stokes parameters. We apply this approach to determine the topological charge of a Laguerre–Gaussian (LG) beam by performing power measurements only. Additionally, we study the connection of the cross-correlation function with the degree of polarization for nonuniformly polarized beams, and we obtain closed-form expressions of the cross correlation for LG vector modes and the generalized full Poincaré beams. © 2014 Optical Society of America

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The potential applications of optical vortices have motivated the development of several techniques to measure the vortex topological charge. The common technique is based on the fork pattern produced when the optical vortex interferes with a reference plane wave [1]. Another interference technique uses the mirror image of the vortex to produce a flower-like pattern where the number of helices is equal to the topological charge [2]. Most recent, diffractive techniques have been demonstrated where the topological charge is inferred from the far-field pattern of a particular diffractive element. Besides the practical application, the diffraction of optical vortices shows interesting effects related to content of orbital angular momentum (OAM). Examples of diffractive techniques are the double slit [3], multipinhole arrays [4], triangular apertures [5,6], annular apertures [7], and axicons [8]. All these techniques require the recording of the complete intensity pattern and the implementation of image processing methods.

In this Letter, we show that it is possible to measure the topological charge by power measurements only. We explore a different approach based on cross-correlation function and Stokes parameter measurements. We study the cross-correlation function $\mathcal{C}(a,b)$ between two complex two-dimensional functions. We show that $\mathcal{C}$ can be measured by the global Stokes parameters. Here, global means the integrated beam intensity over the transverse plane. Then, we show that the autocorrelation function of a Laguerre–Gaussian (LG) beam depends on the topological charge and hence we can use this technique to measure it. The attractive feature of this technique is that it can be easily implemented without the need of a camera. We present a proof of principle experiment to show the viability of this approach. We start with the definition of the cross-correlation function:

$$\mathcal{C}(a,b) = \int f^*(x,y) g(x+a, y+b) \mathrm{d}\mathbf{r}, \tag{1}$$

where the integration is over all the plane, the $*$ means complex conjugate, and a, b are real. In general, $\mathcal{C}(a,b)$ is a complex function. We consider that f and g are solutions to the paraxial wave equation and have unit power, i.e., $\int |f|^2 \mathrm{d}\mathbf{r} = 1$ and $\int |g|^2 \mathrm{d}\mathbf{r} = 1$. Our aim is to show that the cross-correlation function of two optical beams can be obtained by power measurements.

We can express a vector beam as a superposition of two orthogonally polarized components of the form

$$\mathbf{E}(x,y) = E_+(x,y)\hat{\mathbf{c}}_+ + E_-(x,y)\hat{\mathbf{c}}_-, \tag{2}$$

where we have chosen circular polarization basis $\{\hat{\mathbf{c}}_+, \hat{\mathbf{c}}_-\}$ for the polarization states. However, the functions E_+ and E_- are not orthogonal functions in general. We use the centroid of one of the beam components as the optical axis of the system. Then, we translate the centroid of the other component to position (a,b) with respect to the optical axis, and the beam can be written as

$$\mathbf{E}(x,y) = \sqrt{P}[\cos\theta f(x,y)\hat{\mathbf{c}}_+ + \sin\theta g(x+a, y+b)\hat{\mathbf{c}}_-], \tag{3}$$

where P is the total power of the beam, $(x,y) = (0,0)$ correspond to the centroid of the positive circular component and $(x,y) = -(a,b)$ to the centroid of the negative circular component. The parameter θ controls the relative power between both components.

We can show that the global Stokes parameters of $\mathbf{E}(x,y)$ are

$$\begin{aligned} \bar{S}_0 &= P, \\ \bar{S}_1 &= 2P \sin\theta \cos\theta \,\mathrm{Re}\{\mathcal{C}(a,b)\}, \\ \bar{S}_2 &= 2P \sin\theta \cos\theta \,\mathrm{Im}\{\mathcal{C}(a,b)\}, \\ \bar{S}_3 &= P(\cos^2\theta - \sin^2\theta), \end{aligned} \tag{4}$$

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where we have used

$$\bar{S}_i = \int S_i(x, y)\mathrm{d}\mathbf{r}, \quad (5)$$

and S_i is the corresponding Stokes parameter. Then, we can write the cross-correlation function as a function of the global Stokes parameters, that is,

$$\mathcal{C}(a, b) = \frac{\bar{S}_1(a, b) + i\bar{S}_2(a, b)}{(\bar{S}_0^2 - \bar{S}_3^2)^{1/2}}. \quad (6)$$

The previous equation is the first important result of this Letter. This gives a practical method to measure the complex cross-correlation function given in Eq. (1). This means that we can obtain $\mathcal{C}(a, b)$ by measuring the global Stokes parameters as a function of (a, b), which is the centroid of the negative circular component.

Now, we proceed to use the previous result for the determination of the OAM in LG beams. For that, we set $E_+ = E_-$ and therefore $\mathcal{C}(a, b)$ becomes the autocorrelation function $\mathcal{A}(a, b)$.

A normalized LG beam with radial order $p = 0$ and positive azimuthal order m in the waist plane $z = 0$ can be written in the form

$$\psi_m(x, y) = A_m(x + iy)^m \exp(-x^2 - y^2),$$
$$A_m = \left(\frac{2^{m+1}}{\pi m!}\right)^{1/2}, \quad (7)$$

where we have considered the waist to be $w_0 = 1$. We found that the autocorrelation function of ψ_m becomes

$$\mathcal{A}_m(a, b) = \exp\left(-\frac{c^2}{2}\right)\sum_{j=0}^{m} D_j^m c^{2j},$$
$$D_j^m \equiv \frac{1}{\pi m!(-2)^j}\sum_{k=j}^{m}\binom{m}{k}\binom{2k}{2j} \times \Gamma\left(\frac{1}{2} + m - k\right)\Gamma\left(\frac{1}{2} - j + k\right), \quad (8)$$

where $c^2 = a^2 + b^2$ and $\Gamma(\cdot)$ is the Gamma function. We found that $\mathcal{A}_m(a, b)$ is real and an even polynomial of c of order $2m$. Therefore, the roots of this polynomial indicate the azimuthal index of the LG beam.

Assuming an optical power of $P = 1$ and equal-amplitude components, i.e., $\theta = \pi/4$, we find that the global Stokes parameters are

$$\bar{S}_0 = 1, \qquad \bar{S}_1 = \mathcal{A}_m(a, b), \qquad \bar{S}_2 = \bar{S}_3 = 0. \quad (9)$$

Therefore, the azimuthal index and hence the OAM can be determined from $\bar{S}_1(a, b)$. Because of the symmetry, it is possible to set $b = 0$ and find $\mathcal{A}_m$ only as a function of a. Figure 1 shows the resulting polynomials for different orders of m.

We perform the following experiment, shown in Fig. 2. We generate a LG beam by shining a Gaussian laser beam linearly polarized to 45° on a VPP-m633 spiral phase plate

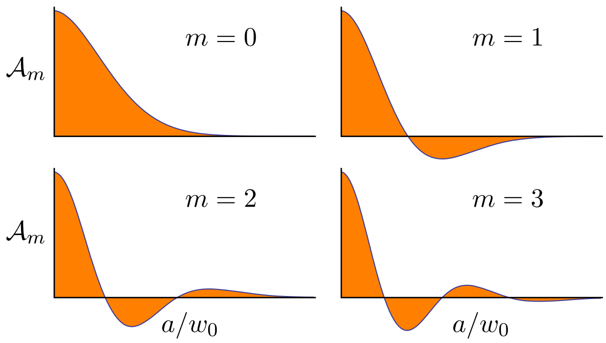

Fig. 1. Autocorrelation function of a LG beam for different azimuthal orders m.

(SPP) from RPC Photonics. The LG beam then enters a Mach–Zehnder (MZ) interferometer through a polarizing beam splitter (PBS), which divides the LG beam in two orthogonally polarized beams, with horizontal and vertical polarizations. We use one quarter-wave plate (QWP) on each interferometer arm to ensure we have a positive circular polarization in one arm and a negative circular polarization in the other. We recombine the now orthogonal circularly polarized beams with a nonpolarizing beam splitter (BS1) to obtain the composite field corresponding to Eq. (2) at the output of the interferometer. The horizontal displacement a between the centroids of the beams is introduced by simultaneously moving the corner mirrors M1 and M2 in the same direction. We move both mirrors simultaneously to avoid introducing a phase due to differences in the optical path. After the MZ interferometer we use a linear or circular polarizer (LP/CP) and a power meter (PM) to measure the global Stokes parameters with

$$\bar{S}_0 = P, \qquad \bar{S}_1 = 2P_x - P,$$
$$\bar{S}_2 = 2P_u - P, \qquad \bar{S}_3 = 2P_+ - P, \quad (10)$$

where P_x, P_u, and P_+ are the optical powers after a horizontal, +45°, and circular polarizer, respectively. The results of the experiment are shown in Fig. 3.

Let us study the relation between the degree of polarization and the cross-correlation function. In practice, the degree of polarization is measured as $d = (\bar{S}_1^2 + \bar{S}_2^2 + \bar{S}_3^2)^{1/2}/\bar{S}_0$. For monochromatic and uniformly polarized fields, the degree of polarization is 1. However, when working with nonuniformly polarized beams the degree of polarization ranges from 0 to 1 [9,10]. For example, for cylindrical vector beams $d = 0$.

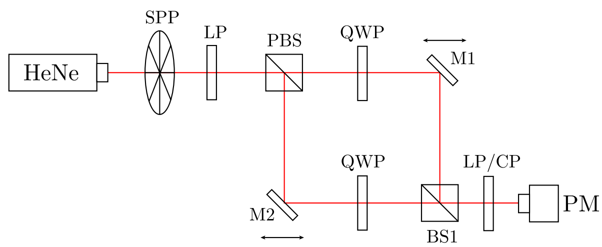

Fig. 2. Experimental setup to observe the autocorrelation function $\mathcal{A}_m(a, 0)$.

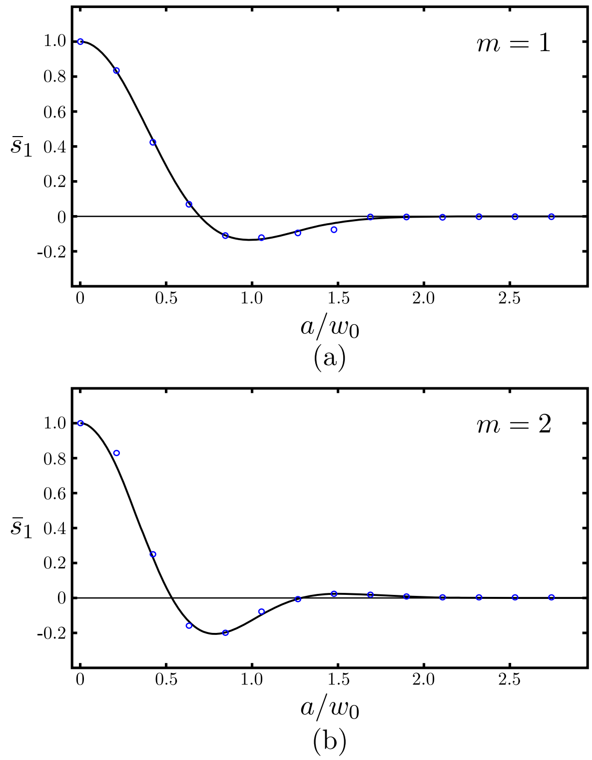

Fig. 3. Global normalized Stokes parameter $\bar{s}_1 = \bar{S}_1/\bar{S}_0$ as a function of the normalized separation a/w_0 for a diagonally polarized LG beam with azimuthal orders (a) $m = 1$ and (b) $m = 2$, theoretical (solid line) and experimental (circles).

Now we consider a superposition of the type in Eq. (2) with equal power, i.e., $\theta = \pi/4$, and therefore, $\bar{S}_3 = 0$. Then, it follows that $d = |\mathcal{C}(a,b)|$, and therefore, for an equal-power superposition we consider $\mathcal{C}(a,b)$ as a *complex* degree of polarization for nonuniformly polarized beams. This can have implications in the development of vector beams that are constructed by an overlap of two orthogonal beams using interferometers [11]. We proceed to study examples of the cross-correlation function and its stability in propagation for particular cases.

Now we derive analytical expressions for the cross-correlation function of different beams formed by superpositions of LG modes. First we obtain closed-form expressions for the cross-correlation function at $z = 0$ for vector modes and generalized Poincaré beams [11]. Vector modes in the circular polarization basis can be constructed as

$$\mathbf{E}_{\mathrm{VM}}(x,y) = \frac{1}{\sqrt{2}}\psi_{-m}(x,y)\hat{\mathbf{c}}_{+} + \frac{1}{\sqrt{2}}\psi_{m}(x,y)\hat{\mathbf{c}}_{-}, \tag{11}$$

and we found that the corresponding cross-correlation function is given by

$$\mathcal{C}_{\mathrm{VM}} = \frac{(-1)^m}{2^m m!}(a+ib)^{2m}\exp\left(-\frac{c^2}{2}\right). \tag{12}$$

Full Poincaré (FP) beams are called as such because the polarization state within the beam spans the entire Poincaré sphere, and the so-called first-order [12], second-order [13], and arbitrary-order [14] FP beams have been studied before. Here we use the general FP beams introduced by Galvez *et al.* in [11], which are written as a superposition LG beams of any two different azimuthal orders m, n with orthogonal circular polarizations,

$$\mathbf{E}_{\mathrm{FP}}(x,y) = \frac{1}{\sqrt{2}}\psi_{m}(x,y)\hat{\mathbf{c}}_{+} + \frac{1}{\sqrt{2}}\psi_{n}(x,y)\hat{\mathbf{c}}_{-}, \tag{13}$$

where we consider the azimuthal orders to be non-negative integers $m, n \geq 0$. Without any loss of generality we assume that $n \geq m$, and we found that the cross-correlation function for such a beam $\mathcal{C}_{m,n}(a,b)$ is given by

$$\mathcal{C}_{m,n} = \frac{(-1)^m}{\sqrt{m!n!}}\exp\left(-\frac{c^2}{2}\right)\left(\frac{a+ib}{\sqrt{2}}\right)^{n-m} \times U\left(-m, 1+(n-m); \frac{c^2}{2}\right), \tag{14}$$

where $U[\gamma,\xi;z]$ is Tricomi's confluent hypergeometric function, which for $\gamma = -m$ is an mth order polynomial of the argument z [15]. If the azimuthal order of the positive circular component is bigger than the one for the negative circular component, i.e., $m > n$, we must include an additional factor $(a^2+b^2)^{n-m}$ in Eq. (14). When $n = m$, the cross-correlation $\mathcal{C}_{m,m}$ becomes the autocorrelation $\mathcal{A}_m$, and Eq. (14) reduces to

$$\mathcal{A}_m = \frac{(-1)^m}{m!}\exp\left(-\frac{c^2}{2}\right)U\left(-m, 1; \frac{c^2}{2}\right), \tag{15}$$

which is a closed-form expression for the autocorrelation function given in Eq. (8). On the other hand, for a standard FP beam ($m = 0, n = 1$), the cross-correlation function simplifies to

$$\mathcal{C}_{0,1} = \frac{1}{\sqrt{2}}(a+ib)\exp\left(-\frac{c^2}{2}\right). \tag{16}$$

Now we obtain the cross-correlation function for beams with complex parameter μ, which allows us to consider the cross correlation under propagation through free-space and ABCD optical systems. The standard LG doughnut beams with input waist w_0 and complex parameter μ can be written as

$$\psi_m(x,y) = \frac{A_m}{(w_0\mu)^{m+1}}(x+iy)^m\exp\left(-\frac{x^2+y^2}{\mu w_0^2}\right), \tag{17}$$

where A_m is the same normalization constant as given in Eq. (7).

For a LG vector mode of angular order m we found that its cross-correlation function is given by

$$\mathcal{C}_{\mathrm{VM}} = \frac{(-1)^m}{m!\alpha}\left(\frac{a+ib}{\sqrt{2}w_0\alpha}\right)^{2m}\exp\left(-\frac{c^2}{2w_0^2\alpha}\right), \tag{18}$$

where $\alpha = \mathrm{Re}\{\mu\}$.

Furthermore, we found that the cross-correlation function for generalized FP beams with complex beam parameter μ and azimuthal orders m, n is given by

$$\mathcal{C}_{m,n} = \frac{(-1)^m}{\sqrt{m!n!}\alpha^{m+1}} \exp\left(-\frac{c^2}{2w_0^2\alpha}\right) \times \left(\frac{a+ib}{\sqrt{2}w_0\alpha}\right)^{n-m} U\left(-m, 1+(n-m); \frac{c^2}{2w_0^2\alpha}\right), \quad (19)$$

where once again we assumed $n \geq m$. However, if $m > n$, we must include an additional factor $(a^2+b^2)^{n-m}$ in Eq. (19). When $n = m$, the cross-correlation $\mathcal{C}_{m,m}$ becomes the autocorrelation $\mathcal{A}_m$, and Eq. (19) reduces to

$$\mathcal{A}_m = \frac{(-1)^m}{m!\alpha^{m+1}} \exp\left(-\frac{c^2}{2w_0^2\alpha}\right) \times U\left(-m, 1; \frac{c^2}{2w_0^2\alpha}\right). \quad (20)$$

For free-space propagation the complex beam parameter is $\mu = 1 + iz/z_R$, where $z_R = kw_0^2/2$ is the Rayleigh length. We can see from Eqs. (18)–(20) that since all the above expressions depend only on the input beam waist w_0 and $\alpha = \mathrm{Re}\{\mu\}$, the cross-correlation function $\mathcal{C}(a,b)$ is invariant under propagation for beams constructed from LG doughnut modes. Furthermore, when the beams propagate through ABCD systems, the cross-correlation function only depends on the real part of the output parameter μ, so we can easily predict how the cross correlation behaves after the beam propagates through such a system.

In conclusion, we proposed the cross-correlation function $\mathcal{C}(a,b)$ as a new way to measure the OAM content of any vector beam. We obtained an expression that allows us to measure the cross correlation from the global Stokes parameters using only power measurements. As an example we obtained an expression for the cross correlation of a diagonally polarized LG beam with radial order $p = 0$ and unknown azimuthal order m, and we found that the now autocorrelation is an even polynomial of order $2m$ of the total displacement $c = \sqrt{a^2+b^2}$ of the beams so that the roots of the polynomial gives us the azimuthal order m of the beam. Furthermore, we found that the cross correlation can be considered as a *complex* degree of polarization for nonuniformly polarized beams, which can have implications in the development of vector beams constructed by overlapping two orthogonal beams using interferometers. In addition to this, we obtained analytic expressions of the cross correlation at $z = 0$ and with complex beam parameter μ for LG vortex modes, as well as generalized FP beams formed by LG beams with different azimuthal orders m, n with orthogonal polarizations. From the expressions of the cross correlations we proved that the cross correlation is invariant under free-space propagation since it only depends on the real part of the complex beam parameter $\mu = 1 + iz/z_R$.

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