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Derivatives of elegant Laguerre–Gaussian beams: vortex structure and orbital angular momentum

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Abstract
The commutation between the paraxial wave equation and the derivative operator allows us to generate novel beam solutions. In this work, we analyze the solutions generated by the derivatives with respect to Cartesian coordinates of elegant Laguerre–Gaussian beams. We present compact expressions for the derivatives of arbitrary integer order and study the resulting orbital angular momentum (OAM) and phase structure. We found that the derivative operator preserves OAM but the topological structure is modified. The resulting topological charge depends on the initial seed beam and the order of the derivative. We also introduce a two-parameter differential operator resulting from the linear combination of Cartesian derivatives $\partial_x$ and $\partial_y$. In analogy with the Poincaré sphere for polarized beams, this operator can be mapped on the surface of a unit sphere. The results can find applications in the generation and control of optical vortex structures.

Keywords: optical vortices, Laguerre–Gaussian beams, vortex arrays

1. Introduction

Elegant Laguerre–Gauss (eLG) beams constitute a biorthogonal complete family of solutions for the paraxial wave equation in cylindrical coordinates [1]. The eLG beams differ from the standard LG beams in that the former contain Laguerre polynomials with a complex argument, whereas in the latter the argument is real and scaled by $\sqrt{2}$. This scaling factor ensures different initial conditions. As a consequence, whereas the standard solutions are shape invariant on propagation (up to a scaling factor), the transverse intensity pattern of the elegant solutions does not remain constant. In any case, the LG beams belong to the class of helical modes with azimuthal dependence $\exp(i l \theta)$, where the integer azimuthal index $l$ characterizes the helical phase structure of the beam.

The consequences of this helical phase are twofold. On the one hand, it suggests the content of orbital angular momentum (OAM) carried by the LG beams [2, 3]. In fact, the LG beams are exploited in multiple applications concerning the transfer of OAM [4]. On the other hand, the helical phase requires a phase singularity at the center of the beam, known as an optical vortex. In general, optical vortices are singular points where the amplitude of the beam is zero and the phase is undefined. The phase twists around the vortices and the number of turns indicates the strength or topological charge of the vortex. The characterization and generation of optical vortices is the core of an extensive field of research known as singular optics [5–7].
In this work we study the Cartesian derivatives of the eLG beams. The motivation is given by the following theorem: consider a solution \( u \) of the paraxial operator \( L \equiv \nabla^2 + i2k \partial / \partial z \); if another linear operator \( D \) commutes with \( L \), the function \( Du \) is also a solution of \( L \). As a consequence, the Cartesian derivatives of the eLG beams are also solutions of the paraxial wave equation.

We show that the \( m \)th order derivative can be constructed as a superposition of eLG beams with the same Gouy shift. This special construction becomes particularly useful for the study of several properties such as OAM and phase singularities. We demonstrate that the Cartesian derivative does not change the content of OAM. However, its phase structure is substantially modified. To generalize the concept of Cartesian derivative, we introduce a two-parameter differential operator resulting from the linear combination of Cartesian derivatives \( \partial_z \) and \( \partial_r \). In analogy with the Poincaré sphere for polarized beams, this operator can be mapped on the surface of a unit sphere. We also study the effect of this operator on the OAM content and vortex structure of the eLG beams. These results can be used in applications for measuring OAM and methods to generate and control optical vortex lattices [8, 9].

2. Derivation rules for elegant Laguerre–Gaussian beams

The complex field amplitude of the monochromatic eLG beam of radial index \( n = (0, 1, 2, \ldots) \) and angular index \( l = (0, \pm 1, \pm 2, \ldots) \), normalized to unit power (i.e. \( \iint |U|^2 \, dx \, dy = 1 \)) and propagating in free space along the positive \( z \) axis of a coordinate system \( \mathbf{r} = (x, y, z) = (r \cos \theta, r \sin \theta, z) \) is given by

\[
U_{n,l}(\mathbf{r}) = \frac{n!}{\sqrt{N!}} \frac{2}{\mu} \left( \frac{r}{\sqrt{\mu w_0}} \right)^{|n|} \times U_{0,0}(\mathbf{r}) L_n^{|l|} \left( \frac{r^2}{\mu w_0^2} \right) \exp(i \theta),
\]

where \( N \equiv 2n + |l| \) is the mode number, \( L_n^{|l|} \) is the associated Laguerre polynomial, \( \mu \equiv \mu(z) = 1 + iz / z_R \) is a longitudinal parameter, \( z_R = k w_0^2 / 2 \) is the Rayleigh distance, \( w_0 \) is the beam width at the waist plane \( z = 0 \), and \( U_{0,0}(\mathbf{r}) \) is the fundamental Gaussian beam

\[
U_{0,0}(\mathbf{r}) = \sqrt{\frac{2}{\pi}} \frac{1}{\mu w_0} \exp \left( - \frac{r^2}{\mu w_0^2} \right). \tag{2}
\]

Elegant Laguerre–Gaussian beams (1) can be constructed from the fundamental Gaussian beam (2) by the repeated application of the ladder operators \( A^+ \) and \( A^- \):

\[
A^\pm \equiv \frac{w_0}{\sqrt{2}} \left( \frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y} \right) = \frac{w_0}{\sqrt{2}} \exp(\pm i \theta) \left( \frac{\partial}{\partial r} \pm i \frac{1}{r} \frac{\partial}{\partial \theta} \right),
\]

as follows:

\[
U_{n,l} = \frac{(-1)^{n+|l|}}{\sqrt{N!}} (A^+)^n (A^-)^{|l|} U_{0,0}, \quad l \geq 0.
\]

We will take advantage of the operator formalism to get closed-form expressions for the derivatives of an eLG beam. The \( m \)th order derivative with respect to \( x \) can be written as a linear combination of the ladder operators, namely

\[
\frac{\partial^m}{\partial x^m} = \left( \frac{A^+ + A^-}{\sqrt{2 w_0}} \right)^m.
\]

By applying this operator to an eLG beam with \( l > 0 \) and using the binomial expansion, we obtain

\[
\frac{\partial^m}{\partial x^m} U_{n,l} = \frac{(-1)^{n+l}}{(\sqrt{2 w_0})^m \sqrt{N!}} \times \sum_{j=0}^m \binom{m}{j} (A^+)^{n+j-l} U_{0,0}.
\]

We observe that the \( m \)th derivative of an eLG can be written as a superposition of \( (m+1) \) constituent eLG modes. Beams with \( j \leq (l + m) / 2 \) have zero or positive azimuthal indices while beams with \( j > (l + m) / 2 \) have negative azimuthal indices. Recalling equations (4) and (5), equation (7) may be rewritten in terms of the constituent beams as follows:

\[
\frac{\partial^m}{\partial x^m} U_{n,l} = C_x \sum_{j=0}^m \binom{m}{j} (-1)^{n+j-l} U_{n',l'}. \tag{8}
\]

where

\[
l' \equiv l + m - 2j,
\]

\[
n' = \begin{cases} n + j, & \text{if } l' \geq 0, \\ n + l + m - j, & \text{if } l' < 0, \end{cases}
\]

\[
C_x = \frac{(-1)^{n+l}}{(\sqrt{2 w_0})^m \sqrt{(N+m)!}} / \sqrt{N!}. \tag{11}
\]

Equation (8) is the first important result of this paper. It shows that the \( m \)th derivative of an eLG beam is composed of \( (m+1) \) beam components \( U_{n',l'} \) with amplitudes equal to the binomial coefficients and indices \( (n', l') \) such that all beam components have the same mode number:

\[
N' = 2n' + |l'| = N + m. \tag{12}
\]

Thus all constituent beams exhibit the same Gouy phase variation along the propagation axis.

In particular, we identify the following rules for the first derivative (see figure 1(a) for a schematic representation):

(i) a beam with indices \( (n, l > 0) \) splits into two beams with indices \( (n, l+1) \) and \( (n+1, l-1) \):

\[
\partial_x U_{n,l>0} = \frac{N+1}{2w_0} (U_{n+1,l-1} - U_{n,l+1}). \tag{13}
\]

(ii) a beam with \( (n, l = 0) \) splits into two beams with indices \( (n, l = 1) \) and \( (n, l = -1) \):

\[
\partial_x U_{n,0} = -\frac{N+1}{2w_0} (U_{n,1} - U_{n,-1}). \tag{14}
\]
(iii) and a beam with indices \((n, l < 0)\) splits into two beams with indices \((n, l - 1)\) and \((n + 1, l + 1)\):

\[
\partial_x U_{n,l=0} = \frac{\sqrt{N+1}}{2w_0^2}(U_{n+1,l=1} - U_{n,l-1}).
\]

These rules can be applied successively to establish a general procedure for obtaining the \(m\)th derivative of an eLG beam. The final structure will depend on the seed beam and the order of the derivative. The recipe is described as follows (see figure 1(b) for a schematic representation of the fourth derivative of \(U_{1,1}\)).

(i) Find the beam components following the above rules. Write down the resulting components in descending order with respect to \(l\). For this example we have \([U_{1,5}, U_{2,3}, U_{3,1}, U_{3,-1}, U_{2,-3}]\). Note that all beam components have mode number \(N' = 7\).

(ii) The coefficients are given by the binomial expansion which for this example are \([1, 4, 6, 4, 1]\).

(iii) Change the sign of the components where \(n' + l'\) is an odd number, see equation (8), which gives us the coefficients \([1, -4, 6, 4, -1]\).

(iv) Calculate the constant \(C_x\) given by equation (11). For this example \(C_x = \sqrt{105/2}w_0^6\).

(v) Finally, combining the previous results we have

\[
\frac{\partial^4 U_{1,1}}{\partial x^4} = \frac{\sqrt{105/2}}{w_0^4}(U_{1,5} - 4U_{2,3} + 6U_{3,1} + 4U_{3,-1} - U_{2,-3}).
\]

We conclude this section by mentioning that following the same procedure used above the \(m\)th derivative of the beam \(U_{n,l}\) with respect to \(y\) is found to be

\[
\frac{\partial^m U_{n,l}}{\partial y^m} = C_y \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} \binom{N+m}{j} U_{n',l'},
\]

\[
C_y = (-i)^m C_x = \frac{(-1)^{n+l} \sqrt{(N+m)!}}{(i\sqrt{2}w_0)^m} \frac{(N+m)!}{N!},
\]

where \(n', l'\), and \(C_x\) are given by equations (9)–(11). Notice that in this case the coefficients of the binomial expansion alternate sign.

We have considered \(l \geq 0\) in the derivation of the equations (8) and (16). However, the cases with negative \(l\) can be determined by taking the complex conjugate of \(\partial^m U_{n,l}\) and using the fact that \(U_{n,l} = U_{n,-l}\). Finally, note that the derivative operator is not a unitary transformation, and therefore the beams (8) and (16) are not normalized to one.

### 3. Generalized Cartesian derivative operator

In section 2 we determined the rules to get the \(m\)th Cartesian derivative of an eLG beam of orders \((n, l)\). In this section we will apply these rules to study the behavior of the OAM and the vortex structure of the derived eLG beams. To proceed in a general way, let us consider the following differential operator resulting from the linear combination of Cartesian derivatives:

\[
\hat{O}_{\alpha,\beta} = \cos \left(\frac{\alpha}{2}\right) \frac{\partial}{\partial x} + \sin \left(\frac{\alpha}{2}\right) \exp(i\beta) \frac{\partial}{\partial y},
\]

where the parameters \(\alpha \in [0, \pi]\) and \(\beta \in (-\pi, \pi]\) control the relative amplitude and phase difference between the derivative operators, respectively. The operator \(\hat{O}_{\alpha,\beta}\) reduces to the Cartesian derivatives \(\partial_x\) and \(\partial_y\) for \((\alpha, \beta) = (0,0)\) and \((\alpha, \beta) = (\pi,0)\), respectively, and is proportional to the ladder operators \(A^\pm\) for \((\alpha, \beta) = (\pi/2, \pm\pi/2)\).

The action of \(\hat{O}_{\alpha,\beta}\) on the beam \(U_{n,l}(\mathbf{r})\) is easily obtained by applying the derivation rules discussed above. We get

\[
W_{n,l}^{\alpha,\beta}(\mathbf{r}) = \frac{w_0}{\sqrt{N+1}} \hat{O}_{\alpha,\beta} U_{n,l}
\]

\[
= \begin{cases} 
C U_{n+1,l-1} - D U_{n,l+1}, & l > 0, \\
C U_{n-1,l} - D U_{n+1,l}, & l = 0, \\
C U_{n,l+1} - D U_{n-1,l}, & l < 0,
\end{cases}
\]

where

\[
C = \frac{1}{\sqrt{2}} \left[ \cos \left(\frac{\alpha}{2}\right) + i \sin \left(\frac{\alpha}{2}\right) \exp(i\beta) \right],
\]

\[
D = \frac{1}{\sqrt{2}} \left[ \cos \left(\frac{\alpha}{2}\right) - i \sin \left(\frac{\alpha}{2}\right) \exp(i\beta) \right],
\]

are complex amplitudes, and the scaling factor \(w_0(N + 1)^{-1/2}\) ensures the normalization and dimensionality of the resulting beam \(W_{n,l}^{\alpha,\beta}(\mathbf{r})\). The operator \(\hat{O}_{\alpha,\beta}\) can be expressed of course as a linear superposition of the ladder operators \(A^\pm\) yielding a helical parameterization of \(\hat{O}_{\alpha,\beta}\). Although there are advantages and disadvantages of each parameterization, neither is more fundamental than the other and both are fully equivalent. In this work, we have chosen the Cartesian basis (i.e. \(\partial_x, \partial_y\)) instead the helical basis (i.e. \(A^+, A^-\)) to emphasize the role of the Cartesian derivative operators.

#### 3.1. Geometrical representation of the operator \(\hat{O}\): the operator sphere

The parameterization of the operator \(\hat{O}_{\alpha,\beta}\) (18) allows one to visualize the values \((\alpha, \beta)\) as the polar \(\alpha\) and azimuthal \(\beta\)
angles of a spherical coordinate system

\[ s = (s_1, s_2, s_3) = (\cos \alpha, \sin \alpha \cos \beta, \sin \alpha \sin \beta) \]  

(22)
on a unit sphere \(|s| = 1\), see figure 2. Thus, in analogy with the concept of the Poincaré sphere for polarization states of light, each point \((\alpha, \beta)\) on the surface of this operator sphere corresponds to a particular value of the operator \(\hat{O}\). Inversely, all possible values of \(\hat{O}\) are uniquely represented on the surface of the operator sphere. As shown in figure 2, the north and south poles of the operator sphere correspond to the Cartesian derivatives \(\partial_x\) and \(\partial_y\) respectively. Points on the circle \(s_1 = 0\) (i.e. \(\beta = \{0, \pi\}\)) represent the directional real derivatives along the unit vector \(\vec{u} = (\cos \alpha/2, \sin \alpha/2)\). The points \(s_3 = \pm 1\) correspond to the ladder operators \(A^\pm\).

For a given seed beam \(U_{n,l}(r)\), the operator \(\hat{O}_{\alpha,\beta}\) maps the beam \(W_{n,l}^{\alpha,\beta}(r)\) on the surface of a beam sphere of order \((n, l)\). Equation (19) can be applied to any eLG beam with indices \((n, l)\), but two examples are of particular relevance: (a) when the seed function is the fundamental Gaussian beam \(U_{0,0}(r)\), and (b) when it is the simplest vortex beam \(U_{0,1}(r)\). Both beam spheres are shown in figures 3 and 4, respectively. Note that figure 3 is isomorphic to the known Poincaré sphere for light beams containing orbital angular momentum, as discussed by several authors [10]. In our case, the modes corresponds to the elegant family of higher-order Gaussian beams and furthermore the operator definition in (19) allows us to extend the concept to higher-order radial and angular indices.

3.2. Orbital angular momentum of the beam \(W_{n,l}^{\alpha,\beta}\)

The \(z\) component of the OAM per photon in unit length about the origin of a transverse slice of a paraxial beam \(U(r)\) reads as

\[ j_z = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |U|^2 \frac{\partial}{\partial \theta} (\arg U) \, dx \, dy, \]  

(23)
where \(j_z\) is in units of \(\hbar\) and \(U(r)\) is power normalized, i.e. \(\int_{-\infty}^{\infty} |U|^2 \, dx \, dy = 1\). It is well known that any vortex beam with angular dependence \(\exp(i\theta)\) carries an intrinsic OAM equal to \(j_z = l\).

We are interested in calculating the OAM of the beam \(\text{eLG}_{n,l}^{\alpha,\beta}\). Replacing (19) into (23) it can be demonstrated that the OAM of \(\text{eLG}_{n,l}^{\alpha,\beta}\) is given by

\[ j_z = l + \sin \alpha \sin \beta = l + s_3. \]  

(24)
Therefore, independently of the indices \((n, l)\) of the seed beam, the operator \(\hat{O}_{\alpha,\beta}\) determines entirely the change in OAM through the distance \(s_1 = \sin \alpha \sin \beta \in [-1, 1]\) along the axis \(A^+ - A^-\) on the operator sphere. This result is illustrated in figure 2 by coloring the surface of the sphere according to the change in OAM. From the figure, it is
clear that the OAM remains unchanged for any Cartesian directional derivative lying on the circle $s_3 = 0$, and reaches its maximum positive and negative changes for the operators $A^+$ and $A^-$ (i.e. when $s_3 = \pm 1$), respectively.

4. Vortex structure

For $l \neq 0$, the helical phase fronts of the eLG beams result in a high-order phase singularity of charge $l$ on the beam axis and the corresponding zero in the intensity pattern. The action of the operator $\hat{O}_{\alpha,\beta}$ on the beam $U_{n,l}(r)$ produces a field $W_{n,l}^{\alpha,\beta}(r)$ with a rich structure of vortices that is fairly complicated, particularly because we need to understand the dependence of the vortex structure on the parameters $(\alpha, \beta)$ and the beam indices $(n, l)$.

4.1. Effect of a directional derivative ($s_3 = 0$)

We first study the effect of a directional derivative represented by any point on the circle $s_3 = 0$ on the operator sphere in figure 2. Without any loss of generality we consider the action of the Cartesian derivative operator $\partial_r$. Evidently, from the azimuthal symmetry of the eLG beams, if the beam is directionally derived along an angle $\phi$ (i.e. $\cos \phi \partial_r + \sin \phi \partial_\theta$), the result is the same arrangement of vortices as obtained with $\partial_r$ but rotated an angle $\phi$.

We begin the analysis by deriving the single ring-shaped eLG beams $U_{0,l}(r)$ with $l > 0$. These beams possess an isolated isotropic vortex of charge $l$ at the origin and do not have rings of phase dislocation. The action of the derivative operator $\partial_{\theta}$ on the beam has the following consequences (see figure 5): (a) the charge of the on-axis vortex decays by one, and (b) two new unit-strength positive vortices are created symmetrically about the origin. By differentiating $m$ times the beam $U_{0,l}(r)$ we can establish the general results.

(i) For the first $m < (l - 1)$ derivatives there is (a) a higher-order vortex with charge $l - m$ at the origin, and (b) $2m$ vortices with charge $+1$ symmetrically located on the $x$ axis.

(ii) For derivatives $m \geq (l - 1)$ there are $l + m$ vortices with charge $+1$ on the $x$ axis.

(iii) The net topological charge is $l + m$.

For any order of the derivative all phase singularities are isotropic, have the same helicity as the original vortex, and form a row of vortices distributed symmetrically around the origin at the roots of the polynomial [11]

$$q(X) = e^{X^2} \frac{d^n}{dX^n} \left(X^2 e^{-X^2}\right),$$

where $X = x/w_0$. Note that for $l = 1$, equation (25) is proportional to the generating function of the Hermite polynomials after the first differentiation, thus we conclude that the $m$th derivative of the simplest vortex beam $U_{0,1}(r)$ has $(m + 1)$ unit vortices located at the roots of the Hermite polynomial $H_{m+1}(X)$.

4.2. Effect of the operator $\hat{O}_{\alpha,\beta}$ with arbitrary parameters $(\alpha, \beta)$

Figure 5 shows the phase structure and the vortex skeleton of the beam $W_{0,0}^{\alpha,\beta} = \hat{O}_{\alpha,\beta}U_{0,0}^{\alpha,\beta}$ at $z = 0$ as the parameters $(\alpha, \beta)$ are varied from the $\beta = -\pi/2$ to $\pi/2$ along the equatorial line $\alpha = \pi/2$.

The action of the derivative $\partial_{\theta}$ on the multiple ring-shaped eLG beams $U_{n,l}(r)$ with radial index $n > 0$ leads to more complex structures of vortices because each phase dislocation ring of the beam $U_{n,l}$ breaks up into two positive vortices on the $x$ axis and two negative vortices on the $y$ axis. Thus, the effect of the radial index $n$ is to add $2n$ positive and $2n$ negative vortices within the vortex structure. Note that the net topological charge of the structure is always $(l + m)$ independently of the radial index $n$.

Figure 6 shows the phase structure and the vortex skeleton of the beam $W_{0,3}^{\alpha,\beta} = \hat{O}_{\alpha,\beta}U_{0,3}^{\alpha,\beta}$ at $z = 0$ as the parameters $(\alpha, \beta)$ are varied from the $\beta = -\pi/2$ to $\pi/2$ along the equatorial line $\alpha = \pi/2$. The seed function is the single-ringed eLG beam $U_{0,3}^{\alpha,\beta}$ with OAM $j_z = 3$ and a third order axial vortex. As shown in the beam sphere, the variation from $\beta = -\pi/2$ to $\pi/2$ represents the continuous up conversion of the eLG beam $U_{1,2}^{\alpha,\beta}$ with OAM $j_z = 2$ into the eLG beam $U_{0,4}^{\alpha,\beta}$ with OAM $j_z = 4$.

At $\beta = -\pi/2$ the field $U_{1,2}$ has an axial vortex of TC $= +2$ and a phase dislocation circle at the overlap of the zero circular contours of the real (red lines) and imaginary (blue lines) parts of the beam. The action of the operator $\hat{O}_{\alpha,\beta}$ with $\beta > -\pi/2$ breaks the circular dislocation and produces the birth of two positive and two negative unitary vortices. As $\beta$ increases, the positive vortices grow closer and the negative get further from the origin. At the critical value $\beta = 0$, the negative vortices disappear and the net TC of the beam changes abruptly from 2 (i.e. $l = 1$) to 4 (i.e. $l + 1$). For positive values of $\beta$ the unitary positive vortices get even closer and finally collapse at the origin creating a fourth-order axial vortex when the beam becomes $U_{0,4}^{\alpha,\beta}$. Note that although the OAM increases continuously as $\beta$ goes from $-\pi/2$ to $\pi/2$, the net topological charge of the beam remains constant and only changes abruptly from 2 to 4 at $\beta = 0$. 

---

**Figure 5.** Decay of the axial higher-order vortex (colored circles) and appearance of unit-strength vortices (white circles) as the Cartesian operator $\partial_{\theta}$ acts successively on a single ringed-shaped eLG beam $U_{0,l}(r)$ with $l = 4$. All vortices are located on the $x$ axis at the zeros of the polynomial (25).
Figure 6. Phase structure and vortex skeleton of the beam $\hat{O}_{\alpha, \beta} U_{0,3}$ as $\beta$ varies in the interval $[-\pi/2, \pi/2]$. Red and blue lines are zero contours of the real and imaginary parts of the field, respectively. Crossing points of these lines give the position of the vortices whose topological charges are represented by white dots $= +1$, black dots $= -1$, green dots $= +2$, and a peach dot $= +4$.

4.3. Mixed derivatives $\partial_q^p \partial_y^p$

Repeated application of the operator $\hat{O}_{\alpha, \beta}$ involves the evaluation of mixed derivatives of the form $\partial_q^p \partial_y^p$ acting on the beam $U_{n,l}$. Mixed derivatives produce interesting array of vortices. Figure 7 shows the intensity and phase structure of the mixed derivatives of the simplest vortex eLG beam $\partial_q^p \partial_y^p U_{0,1}$ for the cases $(q, p)$ equal to $(1, 1), (2, 1)$ and $(2, 2)$. Note that the $x$ derivative produces $1 + p$ columns of positive charges while the $y$ derivative produces $1 + q$ rows of positive charges. The negative charges are contained inside the cells formed by the positive charges.

Mixed derivatives $\partial_q^p \partial_y^p$ acting on higher-order eLG beams generate much more complex structures of vortices. In figure 8 we show the appearance of vortices with different topological charge as the beam $U_{0,4}$ is successively differentiated respect to $x$ and $y$ orders $p$ and $q$, respectively. The table exemplifies the general result that net topological charge of the mixed differentiated beam $\partial_q^p \partial_y^p U_{n,l}$ is $TC = l + p + q$, for any value of the radial index $n$.
4.4. Vortex behavior on propagation

The vortex structures studied above correspond to the initial plane \( z = 0 \). However, contrary to the standard LG beams, the transverse intensity pattern of the eLG beams is not shape invariant on propagation, thus we may anticipate a very complex propagation dynamics of the vortices embedded in the derivatives of the eLG beams.

Figure 9(a) shows the amplitude and phase of the second derivative with respect to \( y \) of the beam \( U_{0,4}(r) \) within the propagation range \( \zeta = z/z_R \in [0, 1] \). The propagation is assumed to be in free space. Applying the derivation rules studied in section 2, it is easy to show that the beam \( \partial_y^2 U_{0,4}(r) \) can be decomposed in terms of the following eLG beams with helicities 2, 4, and 6:

\[
\partial_y^2 U_{0,4} = -\sqrt{15} \frac{2}{w_0^2} (U_2 + 2U_{1,4} + U_{0,6}).
\] (26)

At the initial plane \( z = 0 \), the field is characterized by a row of vortices composed of four unitary positive vortices symmetrically located on the \( y \) axis around a fifth axial vortex of charge +2. As the beam propagates, the unitary vortices rotate around the axial vortex getting further from the central region. As expected, the topological charge and OAM of the beam remain constant at any \( z \) plane.

Figure 9(b) shows the vortex dynamics of a beam with negative vortices. In this case we consider the free-space propagation of the mixed derivative \( \partial_y^2 \partial_x^2 \) acting on the beam \( U_{0,1} \), which can be expanded as

\[
\partial_y^2 \partial_x^2 U_{0,1} = \sqrt{15} \frac{2}{w_0^2} (U_{1,-3} + 2U_{2,1} - U_{0,5}).
\] (27)

Note that the presence of the component \( U_{1,-3} \) with negative helicity leads to the existence of negative vortices in the transverse pattern of \( \partial_y^2 \partial_x^2 U_{0,1} \) at \( z = 0 \). At this plane, the vortex structure is composed of eight positive vortices forming an outer square, four negative vortices at the corners of an inner square, and a ninth axial positive vortex located at the origin. The topological charge of the structure is +5. As the beam propagates, the positive vortices rotate anticlockwise...
and negative vortices clockwise such that at a particular plane ($\zeta \sim 0.3$) the four negative vortices annihilate with the four positive vortices located at the side centers of the outer square. Beyond this plane, the four remaining positive vortices continue rotating and getting further from the origin. In figure 9(b), these vortices are out of the numerical window for the images with $\zeta \geq 0.5$.

5. Conclusions

In summary, we have studied the properties of the Cartesian derivatives of eLG beams. Since the derivative operator commutes with the paraxial operator, these functions are also solutions of the PWE. We found that the $m$th derivative can be expressed as a linear superposition of eLG beams of different mode numbers. Therefore, it is straightforward to study properties such as propagation through $ABCD$ optical systems, orbital angular momentum and phase structure.

We demonstrated that the content of OAM is not changed by a Cartesian derivative. However, the phase configuration is modified depending on the seed beam and the order of the derivative. We have also shown that in free-space propagation, the optical vortices rotate around the beam axis. This is consistent with a rotation of the beam intensity pattern. Different to the case of composite standard LG beams, the rotation is not explained by the Gouy phase difference [12]. In fact, all the component beams have the same Gouy phase. The rotation is attributed to the fact that the eLG are not shape invariant. It would be interesting to study the conditions that could generate the self-imaging or Talbot effect [13].

We studied the properties of the generalized Cartesian derivative operator and calculated the change of OAM. The parameterization of the operator allows one to project the resulting beam on the surface of an operator sphere. This corresponds to the higher-order version of the known Poincaré sphere for OAM states [10]. Therefore, this generalized operator has potential applications in quantum information, as is the case for the first-order Poincaré sphere (for example [14–19]).

As a final remark, we notice that the Cartesian derivatives of eLG beams can find application in the generation of photons in multidimensional vector states of orbital angular momentum [20]. The idea is to prepare photons in a suitable superposition state of normal modes. For example, we can prepare photons in a finite superposition of eLG modes with a specific mean value of the angular momentum per photon. In other words, the order $m$ of the derivative controls the number of states in the superposition while the angular order $l$ of the seed beam controls the content of OAM.

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Appendix A. Orbital angular momentum of the derivative of an eLG beam

In this appendix we will prove that the OAM of the eLG beams is preserved under differentiation with respect to a Cartesian coordinate. As we have seen in the previous sections, every derivative with respect to $x$ of an eLG can be expanded in terms of eLG beams, so here we will do a general calculation for the OAM of an arbitrary superposition of Laguerre-type vortex beams (Bessel beams have the same functional form, so the conclusion drawn here may be extended to a superposition of Bessel beams) and then apply the result to our expansions.

We begin from a complex field of the form

$$ U = \sum_{l} C_{l} F_{l}(r) \exp(il\theta), \quad (A.1) $$

where $F_{l}(r)$ is an arbitrary real function depending only on the radial coordinate and $C_{l}$ is its expansion coefficient. We then proceed to calculate the OAM per photon per unit length of a beam of this type evaluating the expression

$$ j_{z} = \frac{\iint |U|^{2} \frac{\partial}{\partial \theta} \text{arg} U d^{2}r}{\iint |U|^{2} d^{2}r}, \quad (A.2) $$

where $j_{z}$ is in units of $\hbar$. Since $\text{arg} U = \text{arctan}(\text{Im}(U)/\text{Re}(U))$, we may then obtain its derivative with respect to $\theta$ implicitly. We have

$$ \frac{\partial}{\partial \theta} \text{arg} U = \frac{\text{Re} U \frac{\partial}{\partial \theta} \text{Im} U - \text{Im} U \frac{\partial}{\partial \theta} \text{Re} U}{|U|^{2}}. \quad (A.3) $$

The real and imaginary parts of $U$ are easily separated and by differentiating and substituting back into equation (A.2),

$$ j_{z} = \frac{\sum_{l,k} l C_{l} C_{k} \iint F_{l} F_{k} \cos((l-k)\theta) d^{2}r}{\sum_{l,k} C_{l} C_{k} \iint F_{l} F_{k} \exp(i(l-k)\theta) d^{2}r}. \quad (A.4) $$

Evaluation of the $\theta$ integrals yields $2\pi \delta_{l,k}$ for both integrals, so we can set $k = l$ as they are the only nonzero terms.
Additionally, we consider that the components of the expansion carry unit power, i.e., \(2\pi \int_0^\infty F_\ell^2(r) r \, dr = 1\). The previous expression reduces to

\[
j_z = \sum_{l=1}^{m} \frac{j_z C_l^2}{\sum C_l^2}.
\]  

(A.5)

As we can see, the OAM carried by a superposition of vortex beams will be a weighted average of the OAM carried by each of the beams.

Now we can use equation (A.5) to evaluate the OAM of the \(m\)th derivative of an eLG beam given in equation (8). Consider a beam with azimuthal mode number \(l\); the \(m\)th derivative produces a superposition of \(m+1\) eLGs. The \(j\)th component of the expansion contains an OAM equal to \(l + m - 2j\). The coefficients of the component beams are those of the binomial expansion, so, we can write the OAM as

\[
j_z = \frac{\sum_{j=0}^{m}(l + m - 2j) \binom{m}{j}^2}{\sum_{j=0}^{m} \binom{m}{j}^2}.
\]  

(A.6)

Then, we can use the formulas given in section 0.157 of [21],

\[
\sum_{k=0}^{n} k \binom{n}{k}^2 = \frac{(2n - 1)!}{[(n - 1)!]^2},
\]

(A.7)

and demonstrate that \(j_z = l\). Therefore, the OAM is invariant under the \(n\)th Cartesian derivative.

In general, we can prove that the OAM is preserved under mixed derivatives. Consider the field

\[
M_{n,l}^{p,q}(r) = \frac{\partial^p}{\partial y^p} \frac{\partial^q}{\partial x^q} U_{n,l},
\]  

(A.8)

which can be expressed as

\[
M_{n,l}^{p,q} = \frac{C_p^2}{p!} \sum_{j=0}^{p} \frac{q}{2} \sum_{k=0}^{p} \binom{p}{k} (-1)^{k+q} |(-1)^n + |l'|| \times \left( \binom{q}{j} \binom{p}{k} \right) U_{n+l',l'},
\]  

(A.9)

using the equations (8) and (16). The values of \(n''\) and \(l''\) are given with respect to \(n'\) and \(l'\). Now, we can use (A.5) to write

\[
j_z = \sum_{j=0}^{m} \binom{m}{j}^2 \left( \binom{p}{k} \right)^2 \left( \binom{p}{k} \right)^2.
\]  

(A.10)

where \(l' = l + p + q - 2(j + k)\). Finally, using (A.7), it is straightforward to show that \(M_{n,l}^{p,q}\) has OAM of \(j_z = l\).

**Appendix B. Orbital angular momentum of the beam \(W_{n,l}^{\alpha,\beta}\)**

In this appendix we derive equation (24) for the OAM of the beam \(W_{n,l}^{\alpha,\beta}\) produced by the operator \(\hat{O}_{\alpha,\beta}\). We know from (A.2) that

\[
j_z = \iint |W|^2 \frac{\partial}{\partial \theta} (\arg W) \, dx \, dy.
\]  

(B.1)

where we use the fact that \(\iint |W| \, dx \, dy = 1\). We omit the indices of \(W\) for the sake of notation. Then, we can use (A.3) to write

\[
j_z = \iint ReW \frac{\partial}{\partial \theta} ImW \, dx \, dy - \iint ImW \frac{\partial}{\partial \theta} ReW \, dx \, dy.
\]  

(B.2)

Using (19) with \(l > 0\) we have that

\[
2ReW = C U_{n+1,l-1} - D U_{n,l+1}
\]

(B.3)

\[
2iImW = C^* U_{n+1,l-1} - D^* U_{n,l+1}
\]

(B.4)

Additionally, we will use

\[
\frac{\partial}{\partial \theta} U_{n,l} = iU_{n,l+1},
\]  

(B.5)

\[
\frac{\partial}{\partial \theta} U_{n,l}^* = -iU_{n,l-1}^*,
\]  

(B.6)

Gathering the previous equations into (B.2) and using the orthonormality conditions of the eLG beams, we find that

\[
j_z = l(|C|^2 + |D|^2) - |C|^2 + |D|^2.
\]  

(B.7)

Now, we can use the definitions of \(C\) and \(D\) given in (20) and (21) to show that

\[
j_z = l + \sin \alpha \sin \beta.
\]  

(B.8)

The same result is found for values of \(l = 0\) and \(l < 0\).

**References**


