Dynamics of polarization singularities in composite optical vortices

Dorilian Lopez-Mago, Benjamin Perez-Garcia, Adad Yepiz, Raul I Hernandez-Aranda and Julio C Gutierrez-Vega

Photonics and Mathematical Optics Group, Tecnológico de Monterrey, Monterrey, Nuevo León 64849, Mexico

E-mail: dorilian@itesm.mx

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Abstract

We study the dynamics of polarization singularities in the superposition of two off-axis optical vortices with orthogonal polarization states. The motion of C points, states of circular polarization, is described by varying parameters such as separation distance, constant phase difference and topological charge. We analyze the zeros of the Stokes parameters and find analytical solutions for particular cases. The results could be used to realize an alternative method to measure the vortex topological charge.

Keywords: polarization singularities, optical vortices, Laguerre–Gaussian beams

1. Introduction

There is an increasing interest in optical fields with space-variant polarization states. Their topological defects or polarization singularities are attractive because of their stability and conservation properties. These singularities correspond to polarization states where the axes of the polarization ellipse are undefined, i.e. linear and circular polarization states. In a transverse plane, circular polarization states appear as isolated points and are called C points while linear polarization states form lines that are called L lines. In propagation, the C points trace a line and the L lines trace a surface.

Extensive work has been done since Nye’s description of polarization singularities in wavefields [1, 2], which are the vectorial analogies of phase singularities [3, 4]. The classification of these isolated points of circular polarization, according to their morphology, has also been discussed in detail by Dennis [5, 6].

A considerable amount of theoretical and experimental research on the polarization structure of beams has been performed recently, including the study of polarization singularities using q-plates by measuring the Stokes parameters. The tuning of a q-plate allowed Cardano et al [7] to dynamically switch between different polarization states of a beam, for instance between radial and azimuthal polarizations.

The superposition of beams with different transverse profiles, in particular of Laguerre–Gauss and pure Gaussian beams, having orthogonal polarization, has also proven to be a very valuable tool to study the polarization structure and singularities of vector beams, both for fully [8, 9] and partially polarized beams [10]. Beams generated by such a superposition have been named full Poincaré beams due to the fact that they span all the polarization states present on the Poincaré sphere.

Other polarization singularities arising from edge diffraction [11] and from the superposition of random vector waves [12–15] have also been studied, as well as polarization singularities present in paraxial and non-paraxial vector fields [16–19], solitons in nematic liquid crystals [20] and those occurring in the propagation in anisotropic crystals [21–25].

In particular, Flossmann et al [22, 23] analyzed the dynamics of polarization singularities from unfolding an optical vortex using a uniaxial crystal. In their work, a linearly polarized Laguerre–Gaussian beam is refracted using a uniaxial crystal which splits the beam...
into two orthogonal modes. The coherent superposition of both modes leads to a spatial polarization structure containing polarization singularities. It was experimentally demonstrated that the incident optical vortex unfolds into these polarization singularities, providing a connection between phase singularities and polarization singularities with interesting structures of C lines and L surfaces. This unfolding process preserves the topological charge before and during propagation through the crystal.

In the present work we analyze the polarization singularities arising in the superposition of two off-axis orthogonal optical vortex beams. We examine how the polarization singularities are affected by the variation of different control parameters, such as separation distance and topological charges. The motion of $C$ points can be controlled by properly changing the phase and amplitude difference between the two beams, and using this we show how to create and annihilate $C$ points. We characterize the polarization structure based on different morphologies, e.g. according to the inclination angle of the polarization ellipse or the polarization streamlines.

Recently, Luo and Liu [18] derived a solution for the free-space propagation of superimposed Laguerre–Gaussian beams beyond the paraxial approximation. In their work the two Laguerre–Gaussian beams have the same polarization in the initial plane. In this case, the singularities exist due to the longitudinal component of the electric field produced by diffraction. We consider, however, a model where two orthogonally polarized Laguerre–Gaussian beams propagate in parallel but are noncollinear. Therefore, the propagation of the polarization singularities is orthogonal to the transverse plane. This work can be interpreted as the vectorial counterpart of the composite optical vortices studied in the works of Molina-Terriza et al. [26] and that by Maleev and Swartzlander Jr [27].

2. Geometry and mathematical model

In the transverse plane $(x, y)$, a vortex is characterized by the spiral phase $\exp(i m \theta)$, where $\theta$ is the azimuthal angle in polar coordinates. The vortex has a topological charge $m$, that refers to a phase circulation of $2\pi m$ around the point where the singularity occurs, in our case at $x = y = 0$. We use in our model the well known Laguerre–Gaussian beams, with radial order $p = 0$, carrying a vortex $m$ and waist $w_0$ of the form [28]

$$\psi_m(x, y) = (r/w_0)^m \exp(-r^2/w_0^2) \exp(i m \theta),$$  

(1)

where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$.

Let us study the coherent superposition of two off-axis beams of the form in equation (1) with orthogonal polarizations. The composite polarization field can be written as

$$E(x, y) = \exp(i \beta \psi_m(x - x_0, y)\hat{x} + \xi \psi_n(x + x_0, y)\hat{y}.$$  

(2)

This superposition produces a complex space-dependent polarization structure that depends on the separation distance $\Delta x = 2x_0$ between the centroids of the beams and the constant phase difference $\beta$. Variations in the relative amplitude between the two components are allowed by means of the factor $0 \leq \xi \leq 1$. We use the unitary Cartesian polarization vectors $\hat{x}$ and $\hat{y}$. We neglect the effect of diffraction and Gouy phase. In the paraxial regime, this turns out to be a good approximation for describing polarization singularities, as shown in Flossmann et al. [22, 23].

The polarization structure of a vector field is studied using the Stokes parameters expressed in a Cartesian coordinate system

$$S_0 = |E_x|^2 + |E_y|^2, \quad s_1 = \left(|E_x|^2 - |E_y|^2\right)/S_0, \quad s_2 = 2 \text{Re}(E_x^* E_y)/S_0, \quad s_3 = 2 \text{Im}(E_x^* E_y)/S_0,$$

(3)

where $E_x = \exp(i \beta) \psi_m(x - x_0, y)$ and $E_y = \xi \psi_n(x + x_0, y)$. The normalized Stokes parameters $[s_1, s_2, s_3]$ describe a point in the Poincaré sphere that corresponds to a state of polarization. Circular polarization states are located at the poles of the Poincaré sphere. North $[s_1, s_2, s_3] = [0, 0, 1]$ and south $[s_1, s_2, s_3] = [0, 0, -1]$ poles represent, respectively, positive (left-handed) and negative (right-handed) helicity. The equator contains the linearly polarized states and represents the boundary between states with positive or negative helicity.

We focus on polarization singularities known as $C$ points, which correspond to circularly polarized states. There are, however, other types of singularities [19]. $C$ points can be characterized with different morphologies [5]. If we trace the inclination angle of the polarization ellipse given by

$$\alpha = \arg(s_1 + is_2)/2,$$  

(4)

it is observed that its value is undefined in the $C$ points. Around the $C$ point, with a counterclockwise direction, $\alpha$ increases from $-\pi/2$ to $\pi/2$ or decreases from $\pi/2$ to $-\pi/2$, corresponding to a disclination index of $+1/2$ or $-1/2$, respectively. Another classification uses the complex scalar [12]

$$\varphi = E \cdot E,$$  

(5)

whose phase is undefined in the $C$ points. This scalar function contains information about both phase and polarization singularities. In this context, $C$ points carry a topological charge of $\pm 1$. Finally, we have the line classification, which is determined by the pattern of polarization streamlines described in [5].

Our aim is to understand the dynamics of $C$ points with respect to the control parameters $\xi$, $\beta$ and $\Delta x$. For the sake of simplicity the study is restricted to the cases $m = n$ and $m = -n$.

The $C$ points are localized at the intersection of $s_1 = 0$ and $s_2 = 0$, where $s_3 = \pm 1$. We observe from the Stokes parameters in equation (3) that the solutions for $s_1 = 0$ are independent of the phase difference $\beta$ and that the solution for $s_2 = 0$ is independent of the relative amplitude $\xi$. The dynamics of $C$ points is studied by analyzing separately the structure of the solutions for $s_1 = 0$ and $s_2 = 0$. First, we show that the geometry of the $s_1 = 0$ solutions has a characteristic shape which is scaled by the angular order of
the Laguerre–Gaussian beams and is principally determined by the separation distance $\Delta x$. Then, we find analytical solutions for $s_2 = 0$. For the case $m = n$, each set of solutions satisfies the equation of a circle. For $m = -n$ the solutions are hyperbolas.

3. Analysis of the solutions for $s_1 = 0$

The solutions of $s_1 = 0$ correspond to polarization states where the major axis of the polarization ellipse is oriented at $45^\circ$ or $135^\circ$. From equations (2) and (3), we observe that the solutions occur at points where

$$|\psi_m(x-x_0,y)|^2 = \xi^2 |\psi_n(x+x_0,y)|^2$$

is satisfied. Since we consider the cases where $|m| = |n|$, we can assume, without loss of generality, that $m = n = 1$. Therefore, the solutions depend on the separation distance $\Delta x = 2x_0$ and the relative amplitude $\xi$.

Figure 1 shows the solutions for $s_1 = 0$ for different values of $\Delta x$. The first row considers the superposition of beams with equal amplitudes ($\xi = 1$) and in the second row we have two beams with a small difference in their amplitudes ($\xi = 0.99$). The solutions are shown in green, whereas the red points show the centroids of the vortices and the red lines indicate the maximum intensity of the rings, which are given, with respect to each centroid, at a radius of $\rho = w_0/\sqrt{2}$.

For equal-amplitude beams, $s_1 = 0$ contains solutions where the intensities of both beams are equal, which is satisfied at $x = 0$ regardless of $\Delta x$, except for the limiting case $\Delta x = 0$ in which both beams become a single beam with a uniform polarization state. In addition, for $\Delta x > 2\rho$, there is a closed curved around each vortex, due to the nonzero value of the Gaussian envelope of the opposite beam. These closed curves, in combination with the curves of $s_2 = 0$, produce the unfolding of the vortex of charge $m$ into a number of $2m$ polarization singularities, or $C$ points. For $\Delta x < 2\rho$, the two hyperbolas combine with the closed curve of the smallest intensity beam and they become a single open curve. In the extreme case when $\Delta x = 0$, there is a single point solution in $x = y = 0$. This case, with $\xi < 1$, is more likely to be observed experimentally.

Since the solutions are independent of the phase difference $\beta$, the open and close curves, satisfying $s_1 = 0$, become the path of the $C$ points when $\beta$ is varied. In the next section, we present analytical solutions for the equation $s_2 = 0$.

4. Analysis of the solutions for $s_2 = 0$

From the Stokes parameters in equation (3), the solutions of $s_2 = 0$ are independent of the relative amplitude $\xi$. These solutions depend on the topological charges $m$ and $n$, the phase difference $\beta$ and the separation distance $\Delta x$. We focus on two particular cases, when $m = n$ and $m = -n$ with $m > 0$. We found that there are $m$ sets of solutions for each case. For $m = n$ the set of solutions forms circles and for $m = -n$ forms hyperbolas. Details of the solutions are discussed in the appendix.
4.1. Case \( m = n \)

For the case \( m = n > 0 \), we found that there are \( m \) sets of solutions. Each set of solutions, of order \( N \), satisfies the equation of a circle, given by

\[
\begin{align*}
    x^2 + (y + x_0 \tan \Omega y)^2 &= x_0^2 \sec^2 \Omega y, \\
    \Omega y &= \left( \frac{\pi}{2} m - \beta \right), \\
    N &= m - 1, \quad m = 3, \ldots , -(m - 1).
\end{align*}
\]

The circles have different radii and centers, but all circles intersect the centroids of the vortices. Notice that the geometry is similar to a bipolar coordinate system.

The phase \( \beta \) changes the center of the circles along the axis \( y \), and the radius changes in such a way that the circles always pass over the centroids of the beams. For \( \beta = \pi (N - m)/2 \), the circle of order \( N \) becomes the line \( y = 0 \). Figure 2 shows solutions for different values of \( m \) with \( \beta = 0 \).

4.2. Case \( n = -m \)

When the beams contain vortices of opposite charges \( n = -m \), with \( m > 0 \), each set of solutions form hyperbolas given by

\[
\begin{align*}
    x^2 - y^2 &= 2xy \tan \Omega y = x_0^2, \\
    \Omega y &= \left( \frac{\pi}{2} m + \beta \right), \\
    N &= m - 1, \quad m = 3, \ldots , -(m - 1).
\end{align*}
\]

The hyperbolas intersect the centroids of the vortices and the asymptotes cross through \( x = y = 0 \). The asymptotes for a set of index \( N \) are given by \( y = \tan(\pm \pi/4 - \Omega y)/2 \). The phase difference \( \beta \) rotates the asymptotes through an angle \( \beta/2m \). Hence, the asymptotes have an angular frequency of \( 4m\pi \).

5. Dynamics and morphology of \( C \) points

\( C \) points are found in the intersection of the solutions for \( s_1 = 0 \) and \( s_2 = 0 \). Using the results of sections 3 and 4, it is straightforward to infer the motion of \( C \) points with respect to the parameters \( \xi \) and \( \beta \). Let us analyze the case when \( x_0 < \rho \) and \( \xi = 0.99 \), where the solutions of \( s_1 = 0 \) correspond to an open and a closed curve as shown in figure 1. We refer to the open curve as \( O \) and the closed curve as \( U \). By keeping the separation distance constant, the motion of the \( C \) points is controlled by changing \( \beta \). Since the solutions for \( s_1 = 0 \) are independent of \( \beta \), the \( C \) points move along the curves \( O \) and \( U \).

5.1. Case \( m = n \)

Figure 4 shows the motion of \( C \) points as a function of \( \beta \) for the case \( m = n = 2 \). Each \( C \) point returns to its initial...
Figure 4. Dynamics of $C$ points with respect to the phase difference $\beta$ for the case $m = n = 2$. The black and white lines are the solutions of $s_1 = 0$ and $s_2 = 0$, respectively. The intersections of both solutions, the red dots, are the $C$ points. The arrows show the direction of $C$ points if $\beta$ is increased. The diamonds indicate positions where the $C$ points are created or annihilated: (a) $\beta = 0$, (b) $\beta = \pi/3$, (c) $\beta = 2\pi/3$ and (d) $\beta = \pi$.

Figure 5. Morphology of the $C$ points for the case $m = n = 2$: (a) angle $\alpha$ of the polarization ellipse, refer to equation (4), (b) Stokes parameter $s_3$, (c) phase of the complex scalar $\varphi$, defined in equation (5) and (d) polarization streamlines.
where the subscripts $x, y$ indicate spatial derivatives. If $D_L < 0$ the $C$ point is a star, with $D_L > 0$ it is a monstar or a lemon. Then, we distinguish between lemon and monstar with the sign of

$$D_L = [(2s_{1,x} + s_{2,x})^2 - 3s_{2,y}(2s_{1,x} - s_{2,y})] \times [(2s_{1,x} - s_{2,x})^2 + 3s_{2,y}(2s_{1,y} + s_{2,y})] - (2s_{1,x} s_{1,y} + s_{1,x}s_{2,x} - s_{1,y}s_{2,y} + 4s_{2,x}s_{2,y})^2.$$ 

If $D_L > 0$ we have a monstar and if $D_L < 0$ we have a lemon. We have observed that when a particular lemon approaches a star, the lemon turns into a monstar. Therefore, by controlling $\beta$ and $\xi$ we can produce a particular polarization structure.

5.2. Case $n = -m$

Figure 6 shows the motion of $C$ points for the case $m = -n = 2$. Similar to the previous case, the $C$ points move along $O$ from $-\infty$ to $\infty$. However, there are no points of creation or annihilation of $C$ points. In the $U$ curve, the $C$ points rotate with an angular frequency of $2m\pi t$.

The morphology of $C$ points is shown in figure 7. For this case, the total topological charge of $\psi$ is 0. The other morphologies, inclination index and line classification, have similar symmetries to the previous case.

6. Conclusions

We have analyzed the dynamics and morphology of $C$ points in the superposition of two orthogonally polarized and off-axis Laguerre–Gaussian beams. We have shown that the motion of $C$ points can be controlled by varying the phase difference $\beta$ and relative amplitude $\xi$. The solutions of the equations $s_1 = 0$ and $s_2 = 0$ are well defined. For $s_2 = 0$ the solutions describe either circles or hyperbolas depending on whether the composing beams possess topological charges of equal or opposite signs, respectively. It is found that points of circular polarization can be created or annihilated, conserving the total topological charge, by manipulating the phase difference between the two beams.

We have considered that the beams have large Rayleigh distances in order to neglect diffraction effects and the Gouy phase shift. We expect that diffraction would produce a scaling factor in the polarization pattern. Considering beams with the same Rayleigh range $R$, the Gouy phase shift $(|m| + 1)\tan^{-1}(z/R)$ is the same in both beams for the cases that we have analyzed. Therefore, we do not expect that the Gouy phase changes the polarization pattern in these situations. However, for cases where the beams have different topological charges, that is $m \neq n$, the Gouy phase shift could produce a $z$-dependent phase difference, as shown in the study of the full Poincaré beams [8].

Since the number of $C$ points depends on $m$, a potential application of this study is in the measurement of the charge of the vortex beam. Additionally, the sign of the vortex can be measured by the motion of $C$ points with respect to the phase difference. Finally, we emphasize that similar solutions can be found in the superposition of small-core vortices [29].

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Appendix. Derivation of equations (7) and (8)

Case $m = n$

From equation (2) we have that

$$E_x = \exp(i\beta)\psi_m(x - x_0, y),$$
$$E_y = \xi\psi_m(x + x_0, y).$$

(A.1)

It is convenient to express $\psi_m$ in the form

$$\psi_m(x, y) = (x + iy)^m \exp(-x^2 - y^2),$$

(A.2)

where we have considered $w_0 = 1$. We notice that $s_2 = 0$ is equivalent to solving

$$E_x^*E_y + E_xE_y^* = 0.$$  

(A.3)

Substituting equations (A.1) into (A.3) and eliminating common factors, we get

$$\exp(i2\beta)(x^2 + y^2 - x_0^2 + i2x_0y)^m = -\exp(i2\beta)(x^2 + y^2 - x_0^2 + i2x_0y)^m.$$  

(A.4)
Then, we take the $m$th root and the real part, which produces after simplifications

$$x^2 + y^2 - x_0^2 + 2mxy \tan(\Omega_N) = 0,$$

(A.5)

where

$$\Omega_N = \frac{1}{m} \left( \frac{\pi}{2} N - \beta \right), \quad N = -1 - 2\kappa + m,$$

(A.6)

and $\kappa = 0, 1, \ldots, m - 1$ comes from the multiple roots of $(-1)^{1/m}$. Finally, we multiply $x_0^2$ by $\sec^2 \Omega_N - \tan^2 \Omega_N = 1$ to get equation (7)

$$x^2 + (y + x_0 \tan \Omega_N)^2 = x_0^2 \sec^2 \Omega_N.$$

(A.7)

It is straightforward to see from the Stokes parameters in equation (3) that the solutions of $s_3 = 0$ are equal to equation (A.7) with $\beta \to \beta + \pi/2$.

Case $n = -m$

For the case $n = -m$, with $m > 0$, we use

$$E_x = \exp(i\beta)\psi_m(x - x_0, y),$$

$$E_y = \xi \psi_{-m}(x + x_0, y),$$

(A.8)

with

$$\psi_m(x, y) = (x + iy)^m \exp(-x^2 - y^2),$$

$$\psi_{-m}(x, y) = (x - iy)^m \exp(-x^2 - y^2).$$

(A.9)

Similar to the previous case, $E_x^*E_y + E_xE_y^* = 0$ can be expressed as

$$(x^2 - y^2 - x_0^2 - i2xy)^m$$

$$= -\exp(i2\beta)(x^2 - y^2 - x_0^2 + i2xy)^m.$$

(A.10)

We take the $m$th root and the real part to get equation (8)

$$x^2 - y^2 - 2xy \tan(\Omega_N) = x_0^2,$$

$$\Omega_N = \frac{1}{m} \left( \frac{\pi}{2} N + \beta \right),$$

(A.11)

with $N = 1 + 2\kappa - m$ and $\kappa = 0, 1, \ldots, m - 1$. The solutions of $s_3 = 0$ are given by $\beta \to \beta + \pi/2$.

References


[22] Flossmann F, Schwarz U T, Maier M and Dennis M R 2005 Polarization singularities from unfolding an optical vortex through a birefringent crystal Phys. Rev. Lett. 95 253901
[23] Flossmann F, Schwarz U T, Maier M and Dennis M R 2006 Stokes parameters in the unfolding of an optical vortex through a birefringent crystal Opt. Express 14 11402–11