Pancharatnam–Berry phase of optical systems

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We present simple closed-form expressions for evaluating the overall and the Pancharatnam–Berry phase introduced by an optical system with either orthogonal or nonorthogonal eigenpolarizations. The formulas provide a meaningful connection with the Pancharatnam–Berry phase associated with nonclosed paths on the Poincaré sphere.

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It was pointed out separately by Pancharatnam [1] and Berry [2] that if a light beam is taken along a closed cycle in the space of polarization states of light, i.e., the Poincaré sphere, it acquires not only a *dynamic* phase from the accumulated path lengths but also a *geometric* (Pancharatnam–Berry) phase, which is equal to minus half the solid angle subtended by the closed path on the sphere. In his seminal paper [1], Pancharatnam also established a criterion for which two beams with different polarization states are in phase, the so-called *Pancharatnam connection*. As a consequence of this criterion, a Pancharatnam–Berry (PB) phase can also be defined for nonclosed paths on the Poincaré sphere [3–5].

In recent years, there has been increasing interest in (a) vector beams with an inhomogeneous polarization state over a transverse cross section [6,7] and (b) space-variant polarization state manipulators [8,9]. When such beams are passed through optical devices, the fields at different transverse positions traverse different nonclosed paths on the Poincaré sphere, resulting in a space-variant phase front modification that originates from the PB phase.

In this Letter we derive a simple closed-form expression for calculating the PB phase generated by an arbitrary polarization device in terms of its eigenvectors and eigenvalues. The formula applies for a wide class of optical systems characterized by Jones matrices with either orthogonal or nonorthogonal eigenstates and provides a meaningful connection with the PB phase associated with nonclosed paths on the Poincaré sphere.

Consider a Cartesian system in which a coherent vector beam propagates paraxially along the $+z$ axis and the optical elements have plane and parallel surfaces lying in the plane (x, y). The polarization state of the light at a given transverse point of the input face of a polarization device is described by a 2×1 Jones vector:

$$|a\rangle = \begin{bmatrix} a_x \\ a_y \end{bmatrix}, \qquad \begin{matrix} a_x, a_y \in \mathbb{C}, \\ |a_x|^2 + |a_y|^2 = 1, \end{matrix} \tag{1}$$

or, equivalently, by its normalized Stokes vector,

$$\mathbf{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \langle a|\boldsymbol{\sigma}_1|a\rangle \\ \langle a|\boldsymbol{\sigma}_2|a\rangle \\ \langle a|\boldsymbol{\sigma}_3|a\rangle \end{bmatrix} = \begin{bmatrix} |a_x|^2 - |a_y|^2 \\ 2\mathrm{Re}(a_x^* a_y) \\ 2\mathrm{Im}(a_x^* a_y) \end{bmatrix} \in \mathbb{R}^3, \tag{2}$$

where $|\mathbf{A}| = 1$ and $\boldsymbol{\sigma}_j$ are the Pauli spin matrices,

$$\boldsymbol{\sigma}_1 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \boldsymbol{\sigma}_2 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \boldsymbol{\sigma}_3 \equiv \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}. \tag{3}$$

We first assume that, at the same transverse point, the optical element is characterized by two orthonormal eigenpolarization states,

$$|q_1\rangle = \begin{bmatrix} q_x \\ q_y \end{bmatrix}, \qquad |q_2\rangle = \begin{bmatrix} -q_y^* \\ q_x^* \end{bmatrix}, \qquad |q_x|^2 + |q_y|^2 = 1, \tag{4}$$

whose normalized Stokes vectors are $\mathbf{Q}_1 = -\mathbf{Q}_2 = \mathbf{Q} = [Q_1; Q_2; Q_3]$, where $Q_m = \langle q_1|\boldsymbol{\sigma}_m|q_1\rangle$. If a matrix $\mathbf{J}$ has two orthonormal eigenvectors, $|q_1\rangle$ and $|q_2\rangle$, with eigenvalues $\mu_1, \mu_2 \in \mathbb{C}$, then the matrix is

$$\mathbf{J} = \begin{bmatrix} \mu_1|q_x|^2 + \mu_2|q_y|^2 & (\mu_1 - \mu_2)q_x q_y^* \\ (\mu_1 - \mu_2)q_x^* q_y & \mu_2|q_x|^2 + \mu_1|q_y|^2 \end{bmatrix}. \tag{5}$$

The possibility of adjusting the complex parameters (q_x, q_y, μ_1, μ_2) allows one to model a variety of polarization devices, including, for example, ideal and partial polarizers, retardation plates, and polarization rotators. Overall amplitude and phase factors, which are common to any initial state that passes through the optical element, are accounted for in the complex transmittances μ_1, μ_2. We have expressed the Jones vectors and matrices in a Cartesian basis (the usual choice in optics), but it is clear that they may be expressed in any other orthonormal basis, e.g., spinors. In this case, the new Jones matrix would be $\mathbf{LJL}^{-1}$, where $\mathbf{L}$ stands for the linear transformation between the Cartesian and the new basis.

When light $|a\rangle$ passes through the element $\mathbf{J}$, the state of the resulting beam is $|b\rangle = \mathbf{J}|a\rangle$. To compare the input and output states, we recall that, according to the Pancharatnam connection [1,2], the phase difference, ϕ, between two polarization states, $|a\rangle$ and $|b\rangle$, is the phase change, that, when applied to one of them, maximizes the intensity of their superposition. This definition implies that

$$\phi = \arg\langle a|b\rangle = \arg\langle a|\mathbf{J}|a\rangle. \tag{6}$$

Because any complex 2×2 matrix $\mathbf{J} = [j_1, j_2; j_3, j_4]$ may be expanded in a basis of Pauli matrices, we write $\langle a|\mathbf{J}|a\rangle = [(j_1 + j_4)\langle a|\boldsymbol{\sigma}_0|a\rangle + (j_1 - j_4)\langle a|\boldsymbol{\sigma}_1|a\rangle + (j_2 + j_3)\langle a|\boldsymbol{\sigma}_2|a\rangle + \mathrm{i}(j_2 - j_3)\langle a|\boldsymbol{\sigma}_3|a\rangle]/2$.

Replacing the values of j_m from the Jones matrix [Eq. (5)], noting that $\langle a|\boldsymbol{\sigma}_m|a\rangle = A_m$ indeed gives the mth component of the Stokes vector $\mathbf{A}$ [Eq. (2)], and using the definition of the vector $\mathbf{Q}$, we obtain after some algebraic manipulations $\langle a|\mathbf{J}|a\rangle = [\mu_1 + \mu_2 + (\mu_1 - \mu_2)\mathbf{Q}\cdot\mathbf{A}]/2$, where $(\cdot)$ stands for the usual dot product of vectors. In this way, from Eq. (6) the phase difference between the input $|a\rangle$ and output $|b\rangle$ states is

$$\phi = \arg[\mu_1 + \mu_2 + (\mu_1 - \mu_2)\mathbf{Q}\cdot\mathbf{A}]. \tag{7}$$

Equation (7) is the first important result of this Letter. It permits a fast evaluation of the total (dynamic plus geometric) phase change produced by a polarization device with orthogonal eigenvectors in terms of its complex transmittances μ_1, μ_2. One would think that such a nice formula should be well-known, but so far, I have not been able to find it in the optics literature. According to the Pancharatnam connection, state $|a\rangle$ is in phase with state $|b\rangle\exp(-\mathrm{i}\phi)$, or equivalently, the superposition $|a\rangle + |b\rangle\exp(-\mathrm{i}\phi)$ yields maximum intensity.

We now turn our attention to the connection of Eq. (7) with the PB phase. Let $\mathbf{B}$ be the normalized Stokes vector of the output state $|b\rangle = \mathbf{J}|a\rangle$. Using algebraic calculations, we have demonstrated that Eq. (7) may be split into two phases, $\phi = \phi_D + \phi_{PB}$, as follows:

$$\phi = \arg(\det\mathbf{J})/2 + [\Omega_{ABQ} - \Omega_{BA(-Q)}]/4). \tag{8}$$

The first term, $\phi_D = \arg(\det\mathbf{J})/2 = \arg(\mu_1\mu_2)/2$, can be associated to the expected dynamic phase acquired by the beam when it propagates through the optical element.

On the other hand, the second term, $\phi_{PB} = [\Omega_{ABQ} - \Omega_{BA(-Q)}/4]$, corresponds to the PB phase introduced by the element and, as shown in Fig. 1(a), is a quarter of the difference of the areas of the geodesic triangles ABQ and $BA(-Q)$ on the Poincaré sphere, i.e., a quarter of the area of the geodesic quadrangle $ABB^{\dagger}A^{\dagger}$, where $\mathbf{A}^{\dagger} = \mathbf{A} - 2(\mathbf{A}\cdot\mathbf{Q})\mathbf{Q}$. In terms of the Stokes vectors, the triangular areas are

$$\Omega_{ABQ} = 2\arctan\left[\frac{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{Q})}{1 + \mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{Q} + \mathbf{Q}\cdot\mathbf{A}}\right]. \tag{9}$$

Equations (7) and (8) are fully equivalent, and some interesting properties can be directly extracted from them. As shown in Fig. 1(a), the spherical lune formed by the two meridians connecting the antipodal states $\mathbf{Q}$ and $-\mathbf{Q}$ passing through $\mathbf{A}$ and $\mathbf{B}$ has dihedral angle

$$\gamma = \arg\mu_2 - \arg\mu_1 = \arg(\mu_1^*\mu_2) \tag{10}$$

and area 2γ. Therefore, the sum of the areas of the triangles ABQ and $BA(-Q)$ equals 2γ. The value of γ is important because it defines the range of ϕ_{PB}. In Fig. 1(c) we plot the quantity $\mu_1 + \mu_2 + (\mu_1 - \mu_2)\mathbf{Q}\cdot\mathbf{A}$ on the complex plane. Because $\mathbf{Q}\cdot\mathbf{A}\in[-1,1]$, we see that the interval of

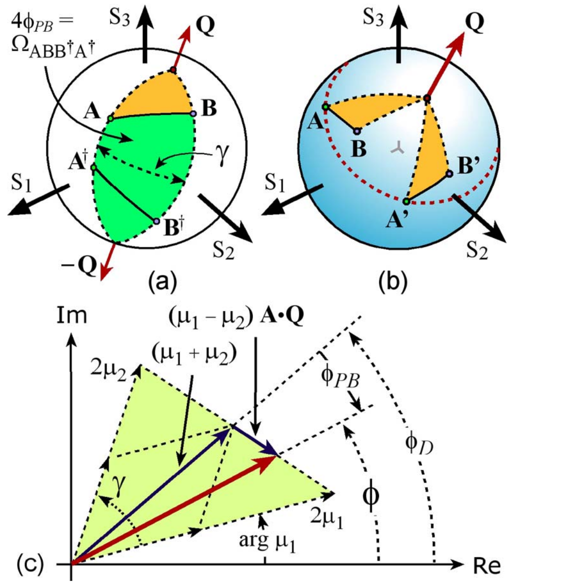

Fig. 1. (Color online) (a) Spherical lune formed by two great circles connecting the orthogonal states $\mathbf{Q}$ and $-\mathbf{Q}$ and the geodesic triangle ABQ on the Poincaré sphere. (b) Curves iso-PB-phase on the Poincaré sphere. (c) Plot of $\mu_1 + \mu_2 + (\mu_1 - \mu_2)\mathbf{Q}\cdot\mathbf{A}$ in the complex plane.

ϕ_{PB} is determined entirely by the phases of the transmittances μ_1, μ_2, namely, $\phi_{PB}\in[-\gamma/2, \gamma/2]$.

Taking $\mathbf{Q}$ to be at a north pole, it follows from Eqs. (7) and (9) that the total and the PB phase remain constant as the input state $\mathbf{A}$ moves along the circular parallels $\mathbf{Q}\cdot\mathbf{A} = \text{const}$, as shown in Fig. 1(b). The triangle ABQ is isosceles only when $|\mu_2/\mu_1| = 1$ and obtuse otherwise. The ratio of arc lengths $\bar{B}Q/\bar{A}Q$ is lower (higher) than unity for values of $|\mu_2/\mu_1|$ lower (higher) than unity.

We will now generalize the results discussed above to a set of N cascaded devices characterized by the matrices $\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_N$. Suppose that the nth element has orthogonal eigenpolarizations $\{\mathbf{Q}_n, -\mathbf{Q}_n\}$ with eigenvalues $\mu_1^{(n)}$, $\mu_2^{(n)}$. Let $|a_0\rangle$ be the Jones vector of the input wave and $|a_n\rangle = \mathbf{J}_n|a_{n-1}\rangle$ the Jones vector as it emerges from the nth element. As illustrated in Fig. 2(a), the sequence of transformations is represented on the Poincaré sphere by the nonclosed path connecting the Stokes vectors $A_0, A_1, \cdots, A_N$ with the shortest geodesic arcs.

The output state of the stack is

$$|a_N\rangle = \mathbf{J}_N\cdots\mathbf{J}_2\mathbf{J}_1|a_0\rangle = \mathbf{M}|a_0\rangle, \tag{11}$$

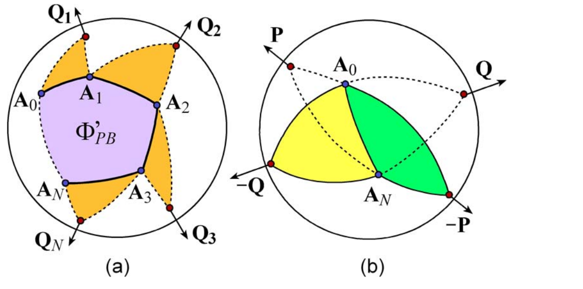

Fig. 2. (Color online) (a) Nonclosed path on the Poincaré sphere for a set of polarization devices. (b) Triangles defined by the initial and final states with the eigenpolarizations $-\mathbf{P}$ and $-\mathbf{Q}$ of the optical system.

where $\mathbf{M} \equiv \mathbf{J}_N \cdots \mathbf{J}_2\mathbf{J}_1$ is the Jones matrix of the optical system. To find the PB phase, Φ_{PB}, introduced by the stack, we first construct a *parallel transported* sequence of transformations, $|a_0\rangle \to |a_1'\rangle \to \cdots |a_N'\rangle$, in which each individual transformation by itself does not change the phase. From Eq. (7), successive states $|a_{n-1}\rangle$, $|a_n\rangle$ are not in phase but differ by a PB phase $\phi_{n-1,n} = \Omega_{A_{n-1}A_nA_n^\dagger A_{n-1}^\dagger}/4$. Multiplying each state $|a_n\rangle$ by the factor $\exp(-\mathrm{i}\phi_{n-1,n})$, factorizing out the exponential terms, and using Eq. (11), we get

$$|a_N'\rangle = \exp\left(-\mathrm{i}\sum_{n=1}^{N}\phi_{n-1,n}\right)|a_N\rangle. \tag{12}$$

Therefore, the actual $|a_N\rangle$ and the parallel transported $|a_N'\rangle$ states differ in phase by the factor in Eq. (12).

We know that if a closed loop is formed by joining the initial and final states with a parallel transported trajectory, then the PB phase Φ_{PB}' is equal to half the solid angle subtended by the enclosed area on the Poincaré sphere [2], which, in our case, is the geodesic polygon $A_0A_1\cdots A_NA_0$ [see Fig. 2(a)]. By triangulating the polygon we get $\Phi_{PB}' = \sum_{n=2}^{N}\phi_{A_nA_{n-1}A_0}$, where $\phi_{A_nA_{n-1}A_0}$ can be determined with Eq. (9).

Now, the actual mechanism by which $|a_0\rangle$ passes successively through the elements cannot be considered parallel transport. Therefore, from Eq. (12) we conclude that the PB phase introduced by the system is

$$\Phi_{PB} = \sum_{n=1}^{N}\phi_{n-1,n} + \sum_{n=2}^{N}\phi_{A_nA_{n-1}A_0}. \tag{13}$$

Equation (13) provides a nice geometrical interpretation of the PB phase generated by the system and holds for simply and multiply connected polygonal paths on the Poincaré sphere.

In analogy to Eq. (7), the total (dynamic plus geometric) phase Φ introduced by the stack can be expressed in terms of its overall eigenpolarizations and eigenvalues. Here the crucial point is that, unlike the eigenvectors of the constituent matrices $\mathbf{J}_n$, the eigenvectors of the overall matrix $\mathbf{M}$ are not orthogonal. Actually, if a system has two nonorthogonal eigenpolarizations $|q\rangle = [q_x; q_y]$, $|p\rangle = [p_x; p_y]$ with Stokes vectors $\{\mathbf{Q}, \mathbf{P}\}$ and eigenvalues $\{\mu_q, \mu_p\}$, then its Jones matrix is

$$\mathbf{M} = \frac{1}{\Delta}\begin{bmatrix} \mu_q q_x p_y - \mu_p p_x q_y & (\mu_p - \mu_q) q_x p_x \\ (\mu_q - \mu_p) q_y p_y & \mu_p q_x p_y - \mu_q p_x q_y \end{bmatrix}, \tag{14}$$

where $\Delta \equiv q_x p_y - p_x q_y \neq 0$.

By applying the same procedure discussed above for deriving Eq. (7), we found that the phase difference Φ between the input and output states of the system can be written as

$$\Phi = \arg\left[\mu_q + \mu_p + (\mu_q - \mu_p)\frac{\langle p^*|\boldsymbol{\sigma}_3(\boldsymbol{\sigma}\cdot\mathbf{A}_0|q\rangle}{\langle p^*|\boldsymbol{\sigma}_3|q\rangle}\right], \tag{15}$$

where $\mathbf{A}_0$ is the Stokes vector of the input wave and $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3)$ is the vector of Pauli matrices. Furthermore, we have demonstrated that phase Φ is related to the half-areas of the triangles $A_0A_N(-P)$ and $A_0A_N(-Q)$ [see Fig. 2(b)] as follows:

$$\Phi = \arg\mu_q + \phi_{A_0A_N(-P)} = \arg\mu_p + \phi_{A_0A_N(-Q)}. \tag{16}$$

Consequently, the area of the polygon $(-P)A_0(-Q)A_N$ is equal to $2\arg(\mu_q^*\mu_p)$. This result generalizes Eq. (10). In the same way, note that Eq. (15) reduces to Eq. (7) when $|q\rangle$ and $|p\rangle$ are orthogonal.

Finally, from Eq. (16) we see that phase Φ splits into a dynamic Φ_D and a PB phase, Φ_{PB}, as follows:

$$\Phi = \arg(\det\mathbf{M})/2 + [\Omega_{A_0A_N(-P)} - \Omega_{A_NA_0(-Q)}]/4, \tag{17}$$

where $\det\mathbf{M} = \mu_p\mu_q = \Pi_n\mu_1^{(n)}\mu_2^{(n)}$ and $\Omega_{A_0A_N(-P)}$ and $\Omega_{A_NA_0(-Q)}$ are the areas of the triangles $A_0A_N(-P)$ and $A_NA_0(-Q)$. Thus, the term $\Phi_{PB} \equiv [\Omega_{A_OA_N}(-P) - \Omega_{A_NA_O}(-Q)]/4$ corresponds to the *PB* phase acquired by the beam from traversing the optical system. Equations (13) and (17) are fully equivalent and reveal a meaningful connection between the area of the polygon $A_0A_1\cdots A_NA_0$ defined by the individual elements of the system [Fig. 2(a)] and the areas of the triangles $A_0A_N(-P)$ and $A_0A_N(-Q)$ defined by the overall matrix of the system [Fig. 2(b)].

In conclusion, we presented simple formulas to calculate the overall and the PB phase introduced by an optical system. These formulas can be easily applied in problems involving the transformation of space-variant polarized beams with polarization manipulators.

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