IOP PUBLISHING JOURNAL OF OPTICS
J. Opt. **12** (2010) 015706 (10pp) doi:10.1088/2040-8978/12/1/015706

Higher-order moments and overlaps of rotationally symmetric beams

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Received 18 August 2009, accepted for publication 25 September 2009
Published 25 November 2009
Online at stacks.iop.org/JOpt/12/015706

Abstract
We introduce a closed-form expression for the overlap between two different circular beams (CiBs) with azimuthal symmetry. A full description of the propagation of the higher-order moments of the CiBs through paraxial ABCD systems is presented. Our formalism can be easily applied to calculate relevant beam parameters such as the normalization constants, the M^2 factors, the kurtosis parameters, the expansion coefficients of the CiBs, and therefore of all its relevant special cases, including the standard, elegant, and generalized Laguerre–Gaussian beams, Bessel–Gaussian beams, hypergeometric–Gaussian beams, quadratic Bessel–Gaussian beams, and optical vortex beams, among others.

Keywords: rotationally symmetric beams, intensity moments, M2 factor, kurtosis, Appell hypergeometric functions

1. Introduction

The 2ν-order overlap integral between the optical beams $U_1(\mathbf{r}_t)$ and $U_2(\mathbf{r}_t)$ is defined by

$$\sigma_{1,2}^{2\nu} = \int_0^{2\pi} \int_0^{\infty} r^{2\nu}\, U_1(\mathbf{r}_t)\, U_2^*(\mathbf{r}_t) r\, \mathrm{d}r\, \mathrm{d}\theta, \qquad (1)$$

where $\mathbf{r}_t = (x, y) = (r, \theta)$ denotes the transverse coordinates, the asterisk means complex conjugate, and ν can be complex in general. The knowledge of explicit expressions for $\sigma_{1,2}^{2\nu}$ is of great importance to characterize and analyze the behavior of known optical beams with circular symmetry. For example, with $\nu = 0$, this integral provides a way to evaluate the coupling efficiency between the fields U_1 and U_2. On the other hand, assuming $U_1 = U_2$, (1) permits us to calculate the 2ν-order intensity moment of the beam.

Recently, circular beams (CiBs) were introduced by Bandres and Gutiérrez-Vega [1] as the general solution of the paraxial wave equation in circular cylindrical coordinates. The complex amplitude of the CiB can be described by either the Whittaker functions or the confluent hypergeometric functions, and are characterized by four independent parameters. Besides the possibility of obtaining novel and meaningful beam structures, one of the most important aspects of the CiBs is that, for special values of their parameters, they reduce to known families of optical beams including the standard, elegant, and generalized Laguerre–Gaussian (LG) beams, the Bessel–Gaussian beams [2, 3], the hypergeometric beams [4], hypergeometric–Gaussian beams [5], the fractional order elegant Laguerre–Gaussian beams [7], the Bessel–Gaussian beams with quadratic radial dependence [8], and the optical vortex beams [9]. Finding general properties of the CiBs is a way to characterize all these special cases at the same time.

In this paper we evaluate the overlap $\sigma_{1,2}^{2\nu}$ between two CiBs with arbitrary parameters. The derivation of $\sigma_{1,2}^{2\nu}$ contributes to the general characterization of the CiBs, and also provides a unified expression for calculating relevant beam parameters such as the higher-order moments, the normalizations, the M^2 factors, the kurtosis parameters, and the expansion coefficients between all the relevant special cases mentioned above.

We also apply the theory of the phase-space distributions to determine the propagation rules of general high-order operators acting on the CiBs and we give explicit expressions for the case of second- and fourth-order moments of these operators. As a further application of our formalism, we obtain the expansion coefficients between the standard and elegant LG beams, that as far as we know have not been reported in the optics literature until now. The results obtained in this work apply to scalar optical beams (i.e. linearly polarized electromagnetic beams) within the standard paraxial regime. Potential applications of the mathematical formalism introduced here are in beam characterization, measurement of laser beam quality, optical metrology, and laser beam shaping and design.

2. Overlap of the circular beams

We begin the analysis by recalling that the transverse distribution at the output plane of a paraxial ABCD system of a CiB with radial $\beta \in \mathbb{C}$ and angular $m = 0, 1, 2, \ldots$ mode numbers is described by the confluent hypergeometric function ${}_1F_1(a, b; x)$ as follows [1]:

$$U_\beta^m(\mathbf{r}_t; q, p) = \zeta (Pr^2)^{m/2}\, {}_1F_1(\beta, m+1; Pr^2) \times \exp\left(\frac{ikr^2}{2q}\right) \exp(\pm i m\theta), \tag{2}$$

where k is the wavenumber and

$$P = P(q, p) = \frac{ik}{2}\left(\frac{1}{p} - \frac{1}{q}\right). \tag{3}$$

The CiBs are characterized by two complex beam parameters (q, p) whose values at the output plane of the system are related to their values (q_0, p_0) at the input plane $(z = 0)$ by the usual bilinear transformations

$$q = \frac{Aq_0 + B}{Cq_0 + D}, \qquad p = \frac{Ap_0 + B}{Cp_0 + D}. \tag{4}$$

In (2), the factor

$$\zeta \equiv \frac{(A + B/p_0)^{(m/2)-\beta}}{(A + B/q_0)^{(m/2)-\beta+1}}, \tag{5}$$

is an overall amplitude factor arising from the propagation of the beam through the ABCD system. The fulfilment of the conditions [1]

$$\mathrm{Im}(1/q_0) > 0, \qquad \mathrm{Im}(1/p_0) > 0, \tag{6}$$

ensures the finiteness of the beam power across the whole transverse plane.

The free space propagation of the CiB along a distance z can be directly obtained from (2) to (5) replacing the values $[A, B; C, D] = [1, z; 0, 1]$. Equation (2) reduces to known families of optical beams with circular symmetry for some special values of the parameters (β, q_0, p_0). For reference purposes, in appendix A we include a table of these special cases.

To evaluate the overlap integral (1) for the CiBs consider the transverse field of two different CiBs at the output plane of the ABCD system

$$U_1 = U_{\beta_1}^{m_1}(\mathbf{r}_t; q_1, p_1), \qquad U_2 = U_{\beta_2}^{m_2}(\mathbf{r}_t; q_2, p_2), \tag{7}$$

where (q_j, p_j), $j = \{1, 2\}$, are given by (4). In considering the fields at the output plane, we are allowing for the possibility that the overlaps intrinsically exhibit their dependence on propagation through paraxial ABCD systems.

For rotationally symmetric beams $U_j(\mathbf{r}_t) = R_j(r)\exp(\pm i m_j\theta)$, the angular integral in (1) can be straightforwardly evaluated and thus the overlap integral reduces to

$$\sigma_{1,2}^{2\nu} = 2\pi\delta_{m_1,m_2}\int_0^\infty r^{2\nu} R_1(r) R_2^*(r)\, r\, dr, \tag{8}$$

where δ_{m_1,m_2} is the Kronecker delta.

By substituting R_1 and R_2 from (2) and making the change of variables $t = r^2$ we get

$$\sigma_{1,2}^{2\nu} = \pi\delta_{m_1,m_2}\zeta_1\zeta_2^*(P_1P_2^*)^{m/2}\int_0^\infty dt\, t^{\nu+m}\exp(-S_{12}t) \times {}_1F_1(\beta_1, m+1; P_1t)\, {}_1F_1(\beta_2^*, m+1; P_2^*t), \tag{9}$$

where $S_{12} \equiv ik(1/q_2^* - 1/q_1)/2$.

Now, using (2.2) of [10] the value of $\sigma_{1,2}^{2\nu}$ turns out to be

$$\sigma_{1,2}^{2\nu} = \pi\delta_{m_1,m_2}\Gamma(\nu+m+1)\zeta_1\zeta_2^*\frac{(P_1P_2^*)^{m/2}}{(S_{12})^{\nu+m+1}} \times F_2\left(\nu+m+1; \beta_1, \beta_2^*; m+1, m+1; \frac{P_1}{S_{12}}, \frac{P_2^*}{S_{12}}\right), \tag{10}$$

where $F_2(a; b, b'; c, c'; x, y)$ is the F_2 Appell hypergeometric function [10–12]. In appendix B we include the definition and a summary of basic properties of the F_2 function. To ensure the convergence of the integral (9), the following conditions: $\mathrm{Re}(\nu + m + 1) > 0$, $\mathrm{Im}(1/q_1 + 1/q_2) > 0$, and $\mathrm{Im}(1/p_1 + 1/p_2) > 0$ must be satisfied. Note that by virtue of (6), the second and third conditions are always fulfilled by CiBs carrying finite power.

Equation (10) is the first important result of this paper. It provides a closed-form expression to evaluate the higher-order overlaps $\sigma_{1,2}^{2\nu}$ between two arbitrary CiBs and also describes its transformation through paraxial ABCD systems.

3. 2ν-order intensity moment $\sigma^{2\nu}$ of the circular beams

As a first application of (10) we obtain the 2ν-order intensity moment $\sigma^{2\nu}$ of the CiBs. By setting $U_1 = U_2 = U_\beta^m$ in (9) and (10), we get

$$\sigma^{2\nu} = \frac{\pi\Gamma(\nu+m+1)|\zeta|^2}{S^{\nu+1}}\left|\frac{P}{S}\right|^m \times F_2\left(\nu+m+1; \beta, \beta^*; m+1, m+1; \frac{P}{S}, \frac{P^*}{S}\right), \tag{11}$$

where

$$S \equiv k\ \mathrm{Im}(1/q), \tag{12}$$

is a positive real quantity. It can be demonstrated using the properties of the Appell function F_2 that, for real values of ν, the evaluation of (11) gives a positive real number, as may be anticipated.

The beam power σ^0 corresponds to the zeroth-order intensity moment and is calculated by setting $\nu = 0$ in (11). Furthermore, the identity (B.9) (i.e. (9.182.3) of [11]) allows us to rewrite σ^0 in terms of the conventional Gaussian hypergeometric function $F = {}_2F_1(a, b; c; x)$ as follows:

$$\sigma^0 = \left| \frac{\pi m!\, \bar{w}^{(m+1)/2}}{\bar{c}^{\,2\beta-m-1}\, \bar{a}} \right| F(\beta, \beta^*; m+1; \bar{w}), \tag{13}$$

where the overlined quantities

$$\bar{c} \equiv \left(1 - \frac{P}{S}\right) \frac{A + B/p_0}{A + B/q_0} = 1 - \frac{P_0}{S_0}, \tag{14a}$$

$$\bar{a} \equiv P(A + B/p_0)(A + B/q_0) = P_0, \tag{14b}$$

$$\bar{w} \equiv \left| \frac{P}{S - P} \right|^2 = \left| \frac{P_0}{S_0 - P_0} \right|^2, \tag{14c}$$

represent quantities whose values remain invariant under propagation through paraxial systems characterized by unimodular ABCD matrices (i.e. $AD - BC = 1$) with real elements. In this way, the recasting of (13) allows us to appreciate immediately the invariance of the beam power on propagation. Once we have obtained the beam power, the normalization constant of the CiBs can be written as $1/\sqrt{\sigma^0}$.

Equation (11) gives the moment $\sigma^{2\nu}$ for any value of ν in terms of the F_2 Appell hypergeometric functions. An alternative procedure to find the even order moments can be obtained by noting in (9) that the derivative of $\sigma_{1,2}^{2\nu}$ with respect to $-S_{12}$ increases by twice the order of the overlap, yielding $\sigma_{1,2}^{2\nu+2}$. Consequently, for the $2n$-order moment we can establish the recursive relation

$$\sigma^{2n} = -\frac{\mathrm{d}\sigma^{2n-2}}{\mathrm{d}S} = \left(-\frac{\mathrm{d}}{\mathrm{d}S}\right)^n \sigma^0, \qquad n = 1, 2, 3, \ldots. \tag{15}$$

In particular, by differentiating σ^0 (13) with respect to $-S$ we obtain the following explicit expression for the second-order intensity moment of the CiBs

$$\sigma_{m,\beta}^2 = -\frac{\mathrm{d}\sigma_{m,\beta}^0}{\mathrm{d}S} = G_\beta^m\, \sigma_{m,\beta}^0 + H_\beta^m\, \sigma_{m+1,\beta+1}^0, \tag{16}$$

where $\sigma_{m,\beta}^0$ and $\sigma_{m+1,\beta+1}^0$ are the powers of the beam U_β^m and the shifted beam $U_{\beta+1}^{m+1}$, respectively, and the factors G_β^m and H_β^m depend on the beam parameters

$$G_\beta^m \equiv \mathrm{Re}\left[\frac{m+1}{S} + \frac{2\beta}{S}\left(\frac{P}{S-P}\right)\right], \tag{17}$$

$$H_\beta^m \equiv \mathrm{Re}\left[\frac{2|\beta|^2}{(m+1)^2} \frac{|P|}{S(P-S)} \left|\frac{A + B/p_0}{A + B/q_0}\right|\right]. \tag{18}$$

The factors G_β^m and H_β^m are a function of the parameters P and S that change on propagation according to (3), (4) and (12). Replacing these expressions into (17) and (18) we can make explicit the dependence of $\sigma_{m,\beta}^2$ on propagation through the ABCD system. After some algebraic arrangements we get

$$G_\beta^m = \mathrm{Re}\left[\frac{m+1}{S_0}\left|A + \frac{B}{q_0}\right|^2 + \frac{2\beta}{S_0}\left(\frac{P_0}{S_0 - P_0}\right)\left(A + \frac{B}{q_0^*}\right)^2\right], \tag{19}$$

$$H_\beta^m = \mathrm{Re}\left[\frac{2|\beta|^2}{(m+1)^2} \frac{\bar{w}\bar{c}}{|P_0|}\left(A + \frac{B}{q_0}\right)\left(A + \frac{B}{p_0^*}\right)\right]. \tag{20}$$

Because all beam parameters involved in (16), (19), and (20) are invariant on propagation, it is clear now that, for free space propagation (i.e. $A = 1$, $B = z$), the second-order moment $\sigma_{m,\beta}^2$ of the CiBs varies always quadratically on z for any value of their parameters.

The iterative application of (16) to evaluate (15) permits us to write the even-th order moments σ^{2n} as a finite summation of the powers of beams with shifted mode numbers. For example, the fourth-order intensity moment can be determined as follows:

$$\sigma_{m,\beta}^4 = -\frac{\mathrm{d}\sigma_{m,\beta}^2}{\mathrm{d}S}, \tag{21a}$$

$$= G_\beta^m \sigma_{m,\beta}^2 - \frac{\mathrm{d}G_\beta^m}{\mathrm{d}S}\sigma_{m,\beta}^0 + H_\beta^m \sigma_{m+1,\beta+1}^2 - \frac{\mathrm{d}H_\beta^m}{\mathrm{d}S}\sigma_{m+1,\beta+1}^0, \tag{21b}$$

$$= \left[(G_\beta^m)^2 - \frac{\mathrm{d}G_\beta^m}{\mathrm{d}S}\right]\sigma_{m,\beta}^0 + \left(G_\beta^m H_\beta^m - \frac{\mathrm{d}H_\beta^m}{\mathrm{d}S} + G_{\beta+1}^{m+1} H_\beta^m\right)\sigma_{m+1,\beta+1}^0 + (H_\beta^m H_{\beta+1}^{m+1})\sigma_{m+2,\beta+2}^0. \tag{21c}$$

This method also preserves the advantage that σ^{2n} can be expressed in terms of the conventional hypergeometric functions $F(a, b; c; x)$.

4. Fourier transform of the circular beams and its 2ν-order moment $\tilde{\sigma}^{2\nu}$

As a second application of (10) we determine the 2ν-order moment of the power spectrum of the CiBs. Let $\mathbf{k}_\mathrm{t} = (k_x, k_y) = (k_\mathrm{t}, \varphi)$ be the transverse position vector in Fourier space. The two-dimensional Fourier transform

$$\tilde{U}(\mathbf{k}_\mathrm{t}) = \frac{1}{2\pi}\int\int_{-\infty}^{\infty} U(\mathbf{r}_\mathrm{t}) \exp(-\mathrm{i}\mathbf{k}_\mathrm{t}\cdot\mathbf{r}_\mathrm{t}) r\, \mathrm{d}r\mathrm{d}\theta, \tag{22}$$

can be obtained by propagating the beam $U(\mathbf{r}_\mathrm{t})$ through a classical $2f$ optical system characterized by the ABCD matrix $[0, k; -1/k, 0]$. Replacing the corresponding values into (2)–(5) we get

$$\tilde{U}_\beta^m(\mathbf{k}_\mathrm{t}) = (\mathrm{i}/k) U_\beta^m(\mathbf{k}_\mathrm{t}; \tilde{q}, \tilde{P}; \tilde{\zeta}), \tag{23}$$

where

$$\tilde{q} \equiv -\frac{k^2}{q}, \qquad \tilde{p} \equiv -\frac{k^2}{p}, \qquad \tilde{P} \equiv \frac{\mathrm{i}(q-p)}{2k}, \tag{24}$$

and

$$\tilde{\zeta} \equiv \frac{(C + D/p_0)^{(m/2)-\beta}}{(C + D/q_0)^{(m/2)-\beta+1}}. \tag{25}$$

Because $U(\mathbf{r})$ and $\tilde{U}(\mathbf{k}_\mathrm{t})$ have the same mathematical structure, (11) can be directly applied to find the propagation of the 2ν-order moment $\tilde{\sigma}^{2\nu}$ of the power spectrum through ABCD systems, so we obtain

$$\tilde{\sigma}^{2\nu} = \sigma^{2\nu}(\tilde{P}, \tilde{S}, \tilde{\zeta})/k^2, \qquad \tilde{S} \equiv -\mathrm{Im}(q)/k. \tag{26}$$

An explicit expression for $\tilde{\sigma}^2$ may also be derived from (16) applying the same parameter transformations and noting that the beam powers in real and Fourier domains are equal by virtue of the Parseval theorem.

5. ABCD formulation for higher-order moments of general operators

In this section we introduce a rigorous ABCD formulation of higher-order moments for rotationally symmetric beams $U(r,\theta) = R(r)\exp(\pm \mathrm{i}m\theta)$ employing a general operator formalism. Let

$$\hat{\mathbf{r}}_\mathrm{t} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}, \qquad \hat{\mathbf{p}}_\mathrm{t} = \frac{1}{\mathrm{i}k}\nabla_\mathrm{t} = \frac{1}{\mathrm{i}k}\left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}}\right) \tag{27}$$

be the transverse position and transverse momentum operators. In terms of $\hat{\mathbf{r}}_\mathrm{t}$ and $\hat{\mathbf{p}}_\mathrm{t}$ we define the operators

$$\hat{\rho} \equiv \hat{\mathbf{r}}_\mathrm{t}\cdot\hat{\mathbf{r}}_\mathrm{t} = x^2 + y^2, \tag{28a}$$

$$\hat{\tau} \equiv \frac{\hat{\mathbf{r}}_\mathrm{t}\cdot\hat{\mathbf{p}}_\mathrm{t} + \hat{\mathbf{p}}_\mathrm{t}\cdot\hat{\mathbf{r}}_\mathrm{t}}{2} = \frac{1}{\mathrm{i}2k}\left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial x}x + y\frac{\partial}{\partial y} + \frac{\partial}{\partial y}y\right), \tag{28b}$$

$$\hat{\kappa} \equiv \hat{\mathbf{p}}_\mathrm{t}\cdot\hat{\mathbf{p}}_\mathrm{t} = -\frac{1}{k^2}\nabla_\mathrm{t}^2 = -\frac{1}{k^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \tag{28c}$$

$$\hat{l}_z \equiv \hat{\mathbf{r}}_\mathrm{t}\times\hat{\mathbf{p}}_\mathrm{t} = \frac{1}{\mathrm{i}k}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right). \tag{28d}$$

Transforming the operators to polar coordinates and taking into account that $\hat{\rho}$, $\hat{\tau}$, and $\hat{\kappa}$ operate over beams with angular dependence $\exp(\pm\mathrm{i}m\theta)$ we get

$$\hat{\rho} = r^2, \tag{29a}$$

$$\hat{\tau} = -\frac{\mathrm{i}}{k}\left(r\frac{\mathrm{d}}{\mathrm{d}r} + 1\right), \tag{29b}$$

$$\hat{\kappa} = -\frac{1}{k^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{m^2}{r^2}\right), \tag{29c}$$

$$\hat{l}_z = \pm m/k. \tag{29d}$$

The commutation relations of $\hat{\rho}$, $\hat{\tau}$, and $\hat{\kappa}$ are given by

$$[\hat{\rho},\hat{\tau}] = (\mathrm{i}2/k)\hat{\rho}, \tag{30a}$$

$$[\hat{\kappa},\hat{\tau}] = -(\mathrm{i}2/k)\hat{\kappa}, \tag{30b}$$

$$[\hat{\kappa},\hat{\rho}] = -(\mathrm{i}4/k)\hat{\tau}. \tag{30c}$$

The operator $\hat{l}_z$ commutes with $\hat{\rho}$, $\hat{\tau}$, and $\hat{\kappa}$.

We now define

$$\langle\hat{\alpha}\rangle = \frac{2\pi}{\sigma^0}\int_0^\infty R^*(\hat{\alpha}R)r\,\mathrm{d}r, \tag{31}$$

as the normalized expectation value of the operator $\hat{\alpha}$ taken with respect to the radial function $R(r)$, where $\sigma^0 = 2\pi\int_0^\infty |R|^2 r\,\mathrm{d}r$ is the power of the beam $U(r,\theta)$.

The physical interpretation of the expectation values of the operators is as follows:

(i) $\langle\hat{\rho}\rangle = \sigma^2/\sigma^0$ is the normalized second-order intensity moment of the beam.

(ii) $\langle\hat{\tau}\rangle$ takes the integral representation $\langle\hat{\tau}\rangle = (2\pi/\sigma^0)\int_0^\infty r\, p_r r\,\mathrm{d}r$ with $p_r(r) = \mathrm{Im}[R^*(\mathrm{d}R/\mathrm{d}r)]$ being the radial component of the momentum density $\mathbf{p} = \mathrm{Im}(U^*\,\nabla U)$ of the beam $U(\mathbf{r}) = R(r)\exp(\mathrm{i}m\theta)$. The value of $\langle\hat{\tau}\rangle$ is related to net momentum flowing into the radial direction at the observation plane. The beam waist corresponds to the condition $\langle\hat{\tau}\rangle = 0$. Positive (negative) values of $\langle\hat{\tau}\rangle$ mean that the beam is diverging (converging) as it propagates in a positive z direction.

(iii) $\langle\hat{\kappa}\rangle k^2 = \tilde{\sigma}^2/\sigma^0$ is the normalized second-order moment of the Fourier transform $\tilde{R}(k_\mathrm{t})$ of the beam, i.e. $\tilde{\sigma}^2 = 2\pi\int_0^\infty k_\mathrm{t}^2|\tilde{R}|^2\,k_\mathrm{t}\,\mathrm{d}k_\mathrm{t}$.

(iv) Finally, $\langle\hat{l}_z\rangle = \pm m/k$ is the z-component of the orbital angular momentum carried by the beam $R(r)\exp(\pm\mathrm{i}m\theta)$. As expected, the value of $\langle\hat{l}_z\rangle$ remains invariant on propagation.

5.1. Propagation of the second-order operators and moments

Beginning from the transformation law for the higher-order operators in Cartesian coordinates that are well known in the theory of the phase-space distributions [13] we have found that the propagation laws of the normalized second-order moments $\langle\hat{\rho}\rangle$, $\langle\hat{\tau}\rangle$, and $\langle\hat{\kappa}\rangle$ through an ABCD system are given by

$$\begin{bmatrix}\langle\hat{\rho}\rangle\\ \langle\hat{\tau}\rangle\\ \langle\hat{\kappa}\rangle\end{bmatrix} = \begin{bmatrix}A^2 & 2AB & B^2\\ AC & AD+BC & BD\\ C^2 & 2CD & D^2\end{bmatrix}\begin{bmatrix}\langle\hat{\rho}\rangle_0\\ \langle\hat{\tau}\rangle_0\\ \langle\hat{\kappa}\rangle_0\end{bmatrix}, \tag{32}$$

where the subscript '0' refers to the corresponding values at the input plane. Equation (32) permits us to evaluate the propagation through a paraxial ABCD system of the second-order moments of the CiBs once their initial values are known. It is important to remark that (32) describes the relations of the operators $\hat{\rho}$, $\hat{\tau}$, and $\hat{\kappa}$, and not only their expectation values. In this case, the operators in the left-hand side act on the beam at the output plane and the operators in the right-hand side act on the beam at the input plane.

Now, to explicitly find the initial values $\langle\hat{\rho}\rangle_0$, $\langle\hat{\tau}\rangle_0$, and $\langle\hat{\kappa}\rangle_0$ in terms of the parameters of the CiBs we take advantage of the fact that (16) already provides the value of $\langle\hat{\rho}\rangle$ at the output plane of an ABCD system. From (32), the transformation rule of $\langle\hat{\rho}\rangle$ also takes the form $\langle\hat{\rho}\rangle = A^2\langle\hat{\rho}\rangle_0 + 2AB\langle\hat{\tau}\rangle_0 + B^2\langle\hat{\kappa}\rangle_0$. By recasting (16) into this transformation rule of $\langle\hat{\rho}\rangle$ we obtain

$$\begin{bmatrix}\langle\hat{\rho}\rangle_0\\ \langle\hat{\tau}\rangle_0\\ \langle\hat{\kappa}\rangle_0\end{bmatrix} = \mathrm{Re}\left\{\begin{bmatrix}1 & 1 & 1\\ 1/q_0^* & 1/q_0^* & (1/q_0 + 1/p_0^*)/2\\ |1/q_0|^2 & 1/q_0^{*2} & 1/q_0 p_0^*\end{bmatrix}\begin{bmatrix}J\\ K\\ L\end{bmatrix}\right\}, \tag{33}$$

where

$$J \equiv \frac{m+1}{S_0}, \tag{34a}$$

$$K \equiv \frac{2\beta P_0}{S_0(S_0 - P_0)}, \tag{34b}$$

$$L \equiv \frac{\beta}{(m+1)^2}\frac{P_0 K^*}{|P_0|}\frac{\sigma^0_{m+1,\beta+1}}{\sigma^0_{m,\beta}}. \tag{34c}$$

Equations (32) and (33) provide the propagation through an ABCD system of the second-order moments of an arbitrary CiB once its beam parameters (m, β, q_0, p_0) are known at a given initial plane.

5.2. Propagation of the fourth-order operators and moments

The formalism introduced above can be also applied for deriving the propagation rules of higher-order operators. For example, the $2n$-order intensity moment $\langle \hat{\rho}^n \rangle = \sigma^{2n}/\sigma^0$ satisfies the transformation rule $\langle \hat{\rho}^n \rangle = \langle (A^2\hat{\rho} + 2AB\hat{\tau} + B^2\hat{\kappa})^n \rangle_0$. By recasting (11) or (15) into this transformation rule for $\langle \hat{\rho}^n \rangle$ we can obtain the values of the unknown expectation values at the initial plane. Alternatively, we can evaluate (11) or (15) at the output plane of $(n+1)(n+2)/2$ different arbitrary ABCD systems and construct a $(n+1)(n+2)/2$-equation system for the unknown initial expectation values.

As an example let us consider the propagation rules of the fourth-order operators. Taking the definition of the operators in (29) we construct the vector of fourth-order moments for the CiBs as follows:

$$\mathbf{W} \equiv \begin{bmatrix} \langle \hat{\rho}^2 \rangle \\ \langle \overline{\rho\tau} \rangle \\ \langle \hat{\epsilon} \rangle \\ \langle \overline{\tau\kappa} \rangle \\ \langle \hat{\kappa}^2 \rangle \end{bmatrix}, \qquad \hat{\epsilon} \equiv (\overline{\rho\kappa} + 2\hat{\tau}^2)/3, \tag{35}$$

where the expectation values are defined as

$$\langle \overline{\rho\tau} \rangle \equiv \langle \hat{\rho}\hat{\tau} + \hat{\tau}\hat{\rho} \rangle/2, \qquad \langle \overline{\rho\kappa} \rangle \equiv \langle \hat{\rho}\hat{\kappa} + \hat{\kappa}\hat{\rho} \rangle/2, \\ \langle \overline{\tau\kappa} \rangle \equiv \langle \hat{\tau}\hat{\kappa} + \hat{\kappa}\hat{\tau} \rangle/2. \tag{36}$$

The propagation of $\mathbf{W}$ through the ABCD system is found to be

$$\mathbf{W} = \begin{bmatrix} A^4 & 4A^3B & 6A^2B^2 & 4AB^3 & B^4 \\ A^3C & A^2(AD+3BC) & 3AB(AD+BC) & B^2(3AD+BC) & B^3D \\ A^2C^2 & 2AC(AD+BC) & A^2D^2+4ABCD+B^2C^2 & 2ABD(AD+BC) & B^2D^2 \\ AC^3 & C^2(3AD+BC) & 3CD(AD+BC) & D^2(AD+3BC) & BD^3 \\ C^4 & 4C^3D & 6C^2D^2 & 4CD^3 & D^4 \end{bmatrix} \mathbf{W}_0, \tag{37}$$

where $\mathbf{W}_0$ denotes the value of the fourth-order moments at the input plane of the system. As occurs for the case of the second-order moments, (37) describes also the relations of the operators and not only of their expectation values. In this case, the operators in the left-hand side act on the beam at the output plane and the operators in the right-hand side act on the beam at the input plane.

$\mathbf{W}_0$ can be determined in terms of the parameters of the CiBs at the input plane by following the same procedure that we used for the second-order moments. From (21*b*) the expression for the propagation of $\langle \hat{\rho}^2 \rangle = \sigma^4_{m,\beta}/\sigma^0_{m,\beta}$ reads as

$$\langle \hat{\rho}^2 \rangle = G^m_\beta \frac{\sigma^2_{m,\beta}}{\sigma^0_{m,\beta}} - \frac{\mathrm{d}G^m_\beta}{\mathrm{d}S} + H^m_\beta \frac{\sigma^2_{m+1,\beta+1}}{\sigma^0_{m,\beta}} - \frac{\mathrm{d}H^m_\beta}{\mathrm{d}S}\frac{\sigma^0_{m+1,\beta+1}}{\sigma^0_{m,\beta}}, \tag{38}$$

where G^m_β and H^m_β are given by (17) and (18), respectively. According to (37), this expression can be recast in the following equivalent form

$$\langle \hat{\rho}^2 \rangle = (\,A^4, \quad 4A^3B, \quad 6A^2B^2, \quad 4AB^3, \quad B^4\,)\mathbf{W}_0. \tag{39}$$

The values of the fourth-order moments are explicitly determined by replacing the second-order moments $\sigma^2_{m,\beta}$ and $\sigma^2_{m+1,\beta+1}$ provided by (32) and (33) in (39). After some algebraic manipulations we obtain

$$\begin{aligned} \mathbf{W}_0 = \mathrm{Re}\Bigg\{ & \boldsymbol{O}_{m,\beta} \begin{bmatrix} J+K \\ (J+K)/q_0^* \\ J/|q_0|^2 + K/q_0^{*2} \end{bmatrix} + \boldsymbol{O}_{m+1,\beta+1} \begin{bmatrix} L \\ L/2q_0 + L/2p_0^* \\ L/q_0 p_0^* \end{bmatrix} \\ & + \frac{\beta}{(S_0 - P_0)^2}\boldsymbol{Y}\left(\frac{1}{p_0}, \frac{1}{q_0^*}\right) - \frac{2\beta - m - 1}{2S_0^2}\boldsymbol{Y}\left(\frac{1}{q_0}, \frac{1}{q_0^*}\right) \\ & - \frac{L}{S_0 - P_0^*}\left[\frac{P_0^*}{S_0}\boldsymbol{T}\left(\frac{1}{q_0}, \frac{1}{p_0^*}\right) - \boldsymbol{Y}\left(\frac{1}{q_0}, \frac{1}{p_0^*}\right)\right]\Bigg\}, \end{aligned} \tag{40}$$

where $\boldsymbol{O}_{m,\beta}$, $\boldsymbol{Y}(u, v)$, and $\boldsymbol{T}(u, v)$ are defined as

$$\boldsymbol{O}_{m,\beta} \equiv \begin{bmatrix} \langle \rho \rangle_0 & 0 & 0 \\ \langle \tau \rangle_0/2 & \langle \rho \rangle_0/2 & 0 \\ \langle \kappa \rangle_0/6 & 2\langle \tau \rangle_0/3 & \langle \rho \rangle_0/6 \\ 0 & \langle \kappa \rangle_0/2 & \langle \tau \rangle_0/2 \\ 0 & 0 & \langle \kappa \rangle_0 \end{bmatrix}_{m,\beta}, \tag{41}$$

$$\boldsymbol{Y}(u, v) \equiv \begin{bmatrix} 2 \\ u+v \\ (u^2+v^2+4uv)/3 \\ uv^2 + u^2v \\ 2u^2v^2 \end{bmatrix}, \tag{42}$$

$$\boldsymbol{T}(u, v) = \begin{bmatrix} 1 \\ (3u+v)/4 \\ (u^2+uv)/2 \\ (u^3+3u^2v)/4 \\ u^3v \end{bmatrix}. \tag{43}$$

Equations (37) and (40) provide the propagation through an ABCD system of the fourth-order moments of an arbitrary CiB once its beam parameters (m, β, q_0, p_0) are known at a given initial plane.

5.3. Invariants

The existence of quantities, named invariants, that are conserved in the course of space propagation along the axis of the paraxial beam is of importance from a theoretical and also a practical point of view [13]. Besides the orbital angular momentum, we identify some basic invariants of the CiBs that come from combinations of the lower-order moments discussed so far.

(i) The simplest invariant is naturally the beam power σ^0 (13).

(ii) In terms of the second-order moments (32), we have the following invariant:

$$\langle\hat{\rho}\rangle\langle\hat{\kappa}\rangle - \langle\hat{\tau}\rangle^2 = \text{const}, \tag{44}$$

that, as we will see later, is directly related to the so-called M^2 quality factor of the beam.

(iii) In terms of the fourth-order moments (35)–(36), we identify the following invariants:

$$\langle\overline{\rho\kappa}\rangle - \langle\hat{\tau}^2\rangle = \text{const}, \tag{45}$$

$$\langle\hat{\rho}^2\rangle\langle\hat{\kappa}^2\rangle - 4\langle\overline{\rho\kappa}\rangle\langle\overline{\tau\kappa}\rangle + 3\langle\hat{\epsilon}\rangle^2 = \text{const}. \tag{46}$$

The physical interpretation of the fourth-order invariants is currently under study by the authors.

6. Characterization parameters and particular examples

Our formalism also allows us to calculate relevant characterization parameters of the CiBs and therefore of all its special cases mentioned in the introduction. In this section we discuss some particular examples.

6.1. M^2 factor

The beam quality factor M^2 is a common parameter characterizing the propagation features of a light beam [14]. In general, for a fixed width at the waist plane a better-quality beam is associated with a lower value of M^2. A minimum value of 1 is reached only for the fundamental Gaussian beam. For rotationally symmetric beams, the M^2 factor is given by

$$M^2 = k\sqrt{\langle\hat{\rho}\rangle\langle\hat{\kappa}\rangle - \langle\hat{\tau}\rangle^2}. \tag{47}$$

Replacing the corresponding values from (32) we can corroborate that M^2 remains invariant for beam propagation through the ABCD system, therefore it may be calculated at the input plane ($z = 0$) using the values in (33).

As a concrete example of application of (47) let us calculate the M^2 factor of the Bessel–Gaussian beams with the quadratic radial dependence introduced by Caron and Potvliege [8] and whose transverse distribution at $z = 0$ is given by

$$\mathrm{QBG}_m(\mathbf{r}_t) \propto J_{m/2}\left(h\frac{r^2}{w_0^2}\right)\exp\left(-\frac{r^2}{w_0^2}\right)\exp(im\theta). \tag{48}$$

For $\{w_0, h\} \in \mathbb{R} > 0$ the waist of the beam (48) coincides with the input plane of the system, therefore the term $\langle\hat{\tau}\rangle$ vanishes and the M^2 factor reduces to $k(\langle\hat{\rho}\rangle\langle\hat{\kappa}\rangle)^{1/2} = (\sigma^2\tilde{\sigma}^2/\sigma^0\tilde{\sigma}^0)^{1/2}$. In this case, the explicit expression for the M^2 factor of the Quadratic Bessel–Gaussian (QBG) beams (48) in terms of the F_2 Appell hypergeometric functions is obtained by replacing the values of the second-order moments (11) and (26) and noting from (A.22) that $P_0/S_0 = \mathrm{i}h$. We obtain

$$M^2 = \bar{m}\sqrt{h^2+1}\,\frac{F_2(\bar{m}+1; \frac{\bar{m}}{2}, \frac{\bar{m}}{2}; \bar{m}, \bar{m}; \mathrm{i}h, -\mathrm{i}h)}{F_2(\bar{m}; \frac{\bar{m}}{2}, \frac{\bar{m}}{2}; \bar{m}, \bar{m}; \mathrm{i}h, -\mathrm{i}h)}. \tag{49}$$

where the short notation $\bar{m} \equiv (m+1)$ has been used for brevity.

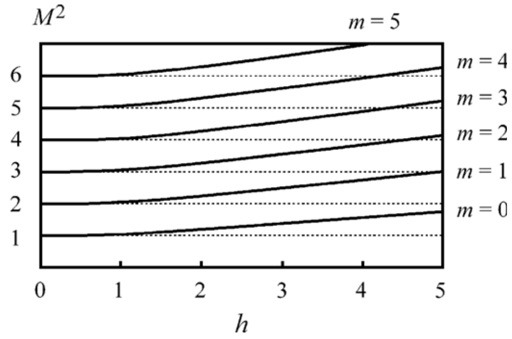

Figure 1. M^2 factor of a Bessel–Gaussian beam with quadratic radial dependence of order m.

Alternatively, the substitution of the parameters (A.22) into (33) and (47) leads to the following expression of M^2 in terms of conventional hypergeometric functions:

$$M^2 = \frac{\bar{m}}{\sqrt{h^2+1}}\left[1 + \frac{h^2}{2} - \frac{h^4}{8}\frac{\bar{m}}{\bar{m}+1} \times \frac{F(\frac{\bar{m}+2}{2}, \frac{\bar{m}+2}{2}; \bar{m}+2; -h^2)}{F(\frac{\bar{m}}{2}, \frac{\bar{m}}{2}; \bar{m}; -h^2)}\right]. \tag{50}$$

Equations (49) and (50) are fully equivalent. Curves of M^2 as a function of h are shown in figure 1 for several values of order m. It can be seen that the quality factor tends to the value $(m + 1)$ when h approaches zero. This result is consistent with the fact that for small values of h the QBG beam (48) is proportional to $r^m\exp(-r^2/w_0^2)\exp(\mathrm{i}m\theta)$, which is indeed proportional to the field of a standard Laguerre–Gaussian (sLG) beam of radial index 0 and azimuthal index m, whose M^2 factor is $(m + 1)$ [15].

6.2. Kurtosis parameter

The kurtosis parameter for rotationally symmetric beams is defined as

$$\mathcal{K} = \frac{\langle\hat{\rho}^2\rangle}{\langle\hat{\rho}\rangle^2} = \frac{\langle r^4\rangle}{\langle r^2\rangle^2} = \frac{(\sigma^4/\sigma^0)}{(\sigma^2/\sigma^0)^2}, \tag{51}$$

and describes the degree of flatness (or sharpness) of the beam intensity distribution. Replacing the corresponding values of the intensity moments from (11) we obtain, after some straightforward simplifications, the general expression of $\mathcal{K}$ in terms of the Appell hypergeometric functions, namely

$$\mathcal{K} = \left(\frac{\bar{m}+1}{\bar{m}}\right) \times \frac{F_2(\bar{m}; \beta, \beta^*; \bar{m}, \bar{m}; \frac{P}{S}, \frac{P^*}{S})F_2(\bar{m}+2; \beta, \beta^*; \bar{m}, \bar{m}; \frac{P}{S}, \frac{P^*}{S})}{[F_2(\bar{m}+1; \beta, \beta^*; \bar{m}, \bar{m}; \frac{P}{S}, \frac{P^*}{S})]^2}, \tag{52}$$

where again $\bar{m} \equiv m + 1$ has been used for brevity.

As a particular example, let us consider the QBG beams (48). Replacing the values of P_0 and S_0 (A.22) into (3) and (12) we get the arguments

$$\frac{P}{S} = \mathrm{i}h\frac{A - (h+\mathrm{i})B/z_R}{A + (h+\mathrm{i})B/z_R}, \qquad \frac{P^*}{S} = \mathrm{i}h\frac{A + (h-\mathrm{i})B/z_R}{A - (h-\mathrm{i})B/z_R}. \tag{53}$$

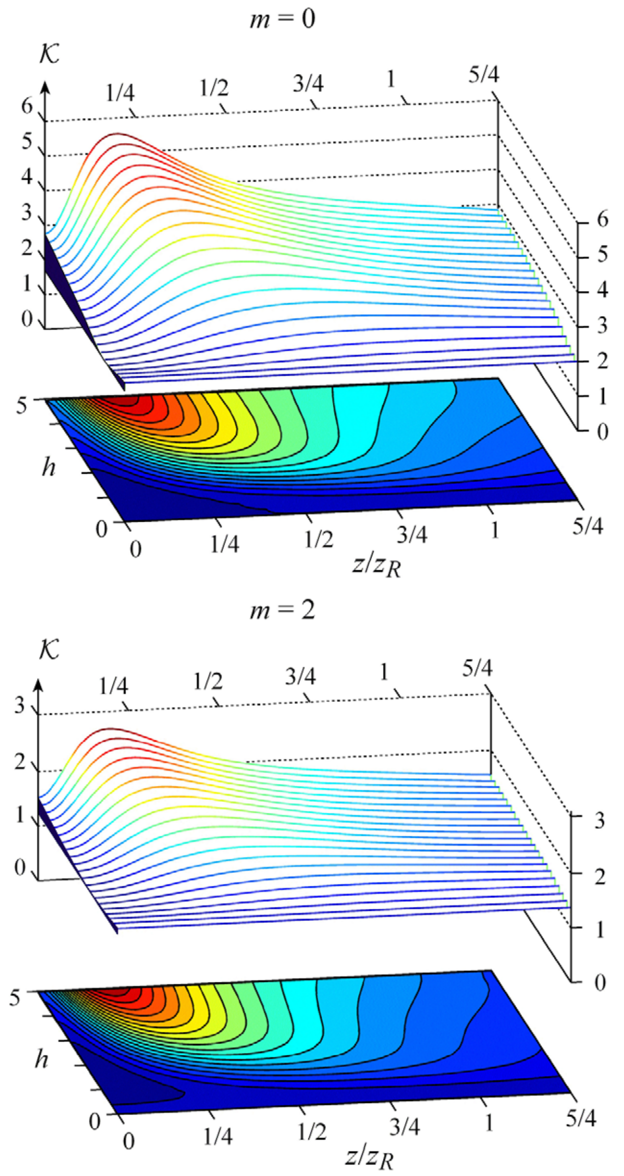

Figure 2. Kurtosis parameter of the QBG beams for $m = 0$ and 2 as a function of the propagation distance and the beam parameter h. (This figure is in colour only in the electronic version)

The kurtosis $\mathcal{K}$ of the QBG beams as a function of the normalized propagation distance $z/z_R \geqslant 0$ and the parameter h is depicted in figure 2 for the orders $m = 0$ and 2. These results were determined evaluating (52) with the arguments (53) for free space propagation, i.e. $A = 1$ and $B = z$. For real values of h, the curves of $\mathcal{K}$ are symmetrical about the plane $z = 0$ and therefore have not been plotted. While M^2 is an invariant quantity for beam propagation through any ABCD optical system, $\mathcal{K}$ changes on propagation except for some special cases. At a given plane z, the beam profile is classified as leptokurtic (sharper profiles), mesokurtic, or platykurtic (flatter profiles) depending on $\mathcal{K}$ being larger, equal to, or less than 2, which is the kurtosis value of the axially symmetric Gaussian beam [16].

For $h = 0$, the kurtosis of the QBG beams becomes constant and equal to $(m + 2)/(m + 1)$; this result is again consistent with the fact that for $h = 0$ the QBG beams are proportional to the sLG_0^m beams, whose $\mathcal{K}$ is $(m + 2)/(m + 1)$ [17]. In figure 2 we can see that, for a given value of h, the kurtosis is minimum at the plane $z = 0$ and has a maximum at a certain propagation distance. As h increases, the location of the maximum gets closer to the plane $z = 0$.

6.3. The embedded Gaussian beam

By calculating the M^2 parameter of a CiB, one can envision an 'embedded' Gaussian beam with a generalized complex radius of curvature Q defined as [14]

$$\frac{1}{Q} \equiv \frac{\langle\hat{\tau}\rangle}{\langle\hat{\rho}\rangle} + \mathrm{i}\frac{M^2/k}{\langle\hat{\rho}\rangle}. \qquad (54)$$

Using (32) we can demonstrate that the transformation Q undergoes when passing through the ABCD system is the conventional bilinear transformation

$$Q = \frac{AQ_0 + B}{CQ_0 + D}. \qquad (55)$$

By rewriting $1/Q = 1/\mathcal{R} + \mathrm{i}2/kw^2$, we determine the equivalent beam radius $w = (2\langle\hat{\rho}\rangle/M^2)^{1/2}$, and the radius of curvature $\mathcal{R} = \langle\hat{\rho}\rangle/\langle\hat{\tau}\rangle$ of the spherical wavefront of the embedded Gaussian. We can carry out beam propagation or design calculations in which we propagate this hypothetical embedded Gaussian through multiple lenses and paraxial elements, finding the focal points and other properties of the embedded Gaussian. Two different beams having similar values of Q can be said to have similar propagation features in practice, in the sense that they spread at the same rate in the far field, and possess the same Rayleigh distance when their transversal extents at their waists are made equal. The waist of the embedded Gaussian is located at the plane where $\langle\hat{\tau}\rangle = 0$.

6.4. Expansion between standard and elegant Laguerre–Gaussian beams

The general expression of the overlap ((10) with $\nu = 0$) provides the coefficients to expand a family of CiBs in terms of other families of CiBs. What we need to do is simply to replace the special values of the parameters (β, q_0, p_0) corresponding to the involved beam families. As a concrete example consider the expansion of the elegant Laguerre–Gaussian (eLG) beams $U_1 = \mathrm{eLG}_n^m(\mathbf{r}_t) = U_{-n}^m(\mathbf{r}_t; q_1, \infty)$, in terms of the orthogonal basis of the standard LG beams $U_2 = \mathrm{sLG}_j^m(\mathbf{r}_t) = U_{-j}^m(\mathbf{r}_t; q_2, q_2^*)$, namely

$$\underline{\mathrm{eLG}_n^m}(\mathbf{r}_t) = \sum\nolimits_{j=0}^{\infty} C_j \underline{\mathrm{sLG}_j^m}(\mathbf{r}_t), \qquad (56)$$

where the underline means that the beams are normalized to unit power. To our best knowledge, the expansion coefficients C_j have not been explicitly reported in the optics literature until now.

For simplicity we calculate the expansion at the waist plane and assume the same waist size w_0 for both eLG and sLG beams, i.e. $q_1 = q_2 = -\mathrm{i}kw_0^2/2$. The coefficients are given by the normalized overlap between the eLG and the sLG beams, namely $C_j = \sigma^0_{1,2}/\sqrt{\sigma^0_1\sigma^0_2}$. Replacing the corresponding values into (10) and (11) we obtain

$$C_j = \frac{F_2(\bar{m}; -n, -j; \bar{m}, \bar{m}; \frac{1}{2}, 1)}{\sqrt{F_2(\bar{m}; -n, -n; \bar{m}, \bar{m}; \frac{1}{2}, \frac{1}{2}) F_2(\bar{m}; -j, -j; \bar{m}, \bar{m}; 1, 1)}}, \tag{57}$$

where $\bar{m} \equiv m + 1$. From (B.3) we know that for negative integer values of its arguments (b, b') the series of the Appell function $F_2(a; b, b'; c, c'; x, y)$ truncates. After some straightforward simplifications we get

$$C_j = \frac{n!(n+m)!}{(n-j)!\sqrt{j!\,(2n+m)!(m+j)!}}, \qquad \text{for } j \leqslant n, \tag{58}$$

and $C_j = 0$ for $j > n$. From (56) and (58) it is found that an eLG beam can be represented as a finite sum of sLG beams and vice versa. Finally, we remark that the norm of the coefficients C_j is unitary, i.e. $\sum_{j=0}^{n} |C_j|^2 = 1$.

7. Conclusions

We have demonstrated in this work that closed-form expressions for the overlap and the 2ν-order intensity moments of the rotationally symmetric circular beams can be written in terms of the Appell hypergeometric function F_2. These expressions allow us to calculate relevant beam parameters such as the higher-order moments, the normalizations, the M^2 factors, the kurtosis parameters, and the expansion coefficients between all the relevant special cases of the CiBs included in the table A.1. It was also possible to determine the propagation rules of the $2n$-order moments of the CiBs through ABCD systems and we gave the explicit expressions for the case of the second-and fourth-order moments. The expansion coefficients between the standard and elegant LG beams, that as far as we know have not been reported in the optics literature until now, were derived as an application of the formalism introduced in this paper.

Acknowledgments

We acknowledge support from Consejo Nacional de Ciencia y Tecnología (grant 82407) and from the Tecnológico de Monterrey (grant CAT-141).

Appendix A. Special cases of the circular beams

For references purposes, in table A.1 we show a list of known special cases of the CiBs and the corresponding values of the beam parameters (β, q_0, p_0) at the initial plane $z = 0$.

The explicit expressions of the field at $z = 0$, the beam powers σ^0, and the M^2 factors for some of the special cases are given below. In all cases, the expressions are obtained by replacing the values included in table A.1 into (2), (13), and (47). The corresponding normalization constant can be calculated with $(1/\sigma^0)^{1/2}$.

(i) Standard Laguerre–Gaussian beam, $\mathrm{sLG}_n^m(\mathbf{r}_t)$

$$P_0 = k\ \mathrm{Im}(1/q_0), \qquad S_0 = P_0, \tag{A.1}$$

$$\mathrm{sLG}_n^m(\mathbf{r}_t) = \frac{m!n!}{(m+n)!}\left[k\,\mathrm{Im}\left(\frac{1}{q_0}\right)r^2\right]^{m/2} \times L_n^m\left[k\ \mathrm{Im}\left(\frac{1}{q_0}\right)r^2\right]\exp\left(\frac{\mathrm{i}kr^2}{2q_0}\right)\exp(\pm \mathrm{i}m\theta), \tag{A.2}$$

$$\sigma^0 = \frac{\pi}{k|\,\mathrm{Im}(1/q_0)|}\frac{(m!)^2 n!}{(m+n)!}, \tag{A.3}$$

$$M^2 = 2n + m + 1. \tag{A.4}$$

(ii) Elegant Laguerre–Gaussian beam $\mathrm{eLG}_n^m(\mathbf{r}_t)$

$$P_0 = -\frac{\mathrm{i}k}{2q_0}, \qquad S_0 = k\,\mathrm{Im}\left(\frac{1}{q_0}\right), \tag{A.5}$$

$$\mathrm{eLG}_n^m(\mathbf{r}_t) = \frac{m!n!}{(m+n)!}\left(-\frac{\mathrm{i}kr^2}{2q_0}\right)^{m/2} \times L_n^m\left(-\frac{\mathrm{i}kr^2}{2q_0}\right)\exp\left(\frac{\mathrm{i}kr^2}{2q_0}\right)\exp(\pm \mathrm{i}m\theta), \tag{A.6}$$

$$\sigma^0 = \frac{2\pi}{k}\frac{(m!)^2(m+2n)!}{[(m+n)!]^2}\left|q_0\left(\frac{1}{2} - \frac{\mathrm{i}}{2}\frac{\mathrm{Re}\ q_0}{\mathrm{Im}\ q_0}\right)^{2n+m+1}\right|, \tag{A.7}$$

$$M^2 = \sqrt{(2n+m+1)\left(\frac{2n+m+m^2}{2n+m}\right)}. \tag{A.8}$$

For fractional elegant Laguerre–Gaussian beams fr-$\mathrm{eLG}_n^m(\mathbf{r}_t)$, [7] replace $n = 0, 1, 2, \ldots$ by $\eta \in \mathbb{R}$.

(iii) generalized Laguerre–Gaussian (gLG) beam, $\mathrm{gLG}_n^m(\mathbf{r}_t)$

$$P_0 = \frac{\mathrm{i}k}{2}\left(\frac{1}{p_0} - \frac{1}{q_0}\right), \qquad S_0 = k\,\mathrm{Im}(1/q_0), \tag{A.9}$$

$$\mathrm{gLG}_n^m(\mathbf{r}_t) = \frac{m!n!}{(m+n)!}(P_0 r^2)^{m/2} L_n^m(P_0 r^2) \times \exp\left(\frac{\mathrm{i}kr^2}{2q_0}\right)\exp(\pm \mathrm{i}m\theta), \tag{A.10}$$

$$\sigma^0 = \frac{\pi m!}{k\,\mathrm{Im}(1/q_0)}\left|\left(\frac{P_0}{S_0}\right)^m\left(\frac{S_0 - P_0}{S_0}\right)^{2n}\right| \times F\left(-n, -n; m+1; \left|\frac{P_0}{S_0 - P_0}\right|^2\right). \tag{A.11}$$

(iv) Bessel–Gaussian (BG) beams, $\mathrm{BG}_{k_t}^m(\mathbf{r}_t)$ [2, 3],

$$P_0 = k_t^2/4n, \qquad S_0 = k\,\mathrm{Im}(1/q_0), \tag{A.12}$$

$$\mathrm{BG}_{k_t}^m(\mathbf{r}_t) = \left[\frac{m!\,n!\,n^{m/2}}{(n+m)!}\right] J_m(k_t r) \times \exp\left(\frac{\mathrm{i}kr^2}{2q_0}\right)\exp(\pm \mathrm{i}m\theta), \tag{A.13}$$

$$\sigma^0 = \left[\frac{m!\,n!\,n^{m/2}}{(n+m)!}\right]^2 \frac{\pi}{k\ \mathrm{Im}(1/q_0)}\frac{\mathrm{I}_m(\alpha)}{\exp(\alpha)}, \tag{A.14}$$

$$M^2 = \sqrt{\left[m + 1 + \alpha\frac{\mathrm{I}_{m+1}(\alpha)}{\mathrm{I}_m(\alpha)}\right]^2 - \alpha^2}, \tag{A.15}$$

where I_m is the modified Bessel function of the first kind and $\alpha = k_t^2/[2k\ \mathrm{Im}(1/q_0)]$ for $k_t \in \mathbb{R}$.

Table A.1. Special cases of $U_\beta^m(\mathbf{r}; q_0, p_0)$.

Special cases	β	q_0	p_0	Comment
sLG_n^m	$-n$	q_0	q_0^*	$n = 0, 1, 2, \ldots$
eLG_n^m	$-n$	q_0	∞	$n = 0, 1, 2, \ldots$
gLG_n^m	$-n$	q_0	p_0	$n = 0, 1, 2, \ldots$
HyGG_μ^m [5]	$-\mu/2$	q_0	0	$\mu \in \mathbb{R}$ (for $\mu \in \mathbb{C}$ see [6])
fr-eLG_η^m [7]	$-\eta$	q_0	∞	$\eta \in \mathbb{R}$
$\mathrm{BG}_{k_t}^m$ [2]	$-n$	q_0	$(q_0^{-1} - \mathrm{i}k_t^2/2nk)^{-1}$	$\lim_{n\to\infty} U \propto J_m(k_t r)\exp(\frac{ikr^2}{2q_0})$
VB^m [9]	$m/2$	∞	0	$\mathrm{VB}^m(r, \theta, 0) \propto \exp(im\theta)$
HyG_β^m [4]	β	∞	0	
WGB_β^m [18]	β	$-p_0^*$	$(kw_0^2/2)/(h+\mathrm{i})$	$\{w_0, h\} \in \mathbb{R}$
QBG^m [8]	$(m+1)/2$	$-p_0^*$	$(kw_0^2/2)/(h+\mathrm{i})$	$\{w_0, h\} \in \mathbb{R}$

(v) Whittaker–Gaussian beams, $\mathrm{WGB}_\beta^m(\mathbf{r}_t)$, [18]

$$P_0 = \mathrm{i}h2/w_0^2, \qquad S_0 = 2/w_0^2, \tag{A.16}$$

$$\mathrm{WGB}_\beta^m(\mathbf{r}_t) = \left(2h\frac{\mathrm{i}r^2}{w_0^2}\right)^{-1/2} M_{(m+1-2\beta)/2,m/2}\left(2h\frac{\mathrm{i}r^2}{w_0^2}\right) \times \exp\left(-\frac{r^2}{w_0^2}\right)\exp(\pm im\theta), \tag{A.17}$$

$$\sigma^0 = \frac{\pi m! h^m w_0^2}{2|(1-\mathrm{i}h)^{2\beta}|} F\left(\beta, \beta^*; m+1; \frac{h^2}{1+h^2}\right), \tag{A.18}$$

$$M^2 = \frac{1}{4}\sqrt{\frac{u^2 - 16h^2(\mathrm{Re}\mu)^2 - 16|\mu|^2}{1+h^{-2}}}, \tag{A.19}$$

where $\mu = 1 + m - 2\beta$, and

$$u \equiv 4h\mathrm{Re}(\mu) + \frac{4(1+m)}{h} + \frac{8|\beta|^2}{(1+m)^2}\frac{\sigma^0_{m+1,\beta+1}}{\sigma^0_{m,\beta}}. \tag{A.20}$$

(vi) Bessel–Gaussian beams with quadratic radial dependence, $\mathrm{QBG}^m(\mathbf{r}_t)$ [8],

$$P_0 = \mathrm{i}h2/w_0^2, \qquad S_0 = 2/w_0^2, \tag{A.21}$$

$$\mathrm{QBG}^m(\mathbf{r}_t) = (\mathrm{i}4)^{m/2}\Gamma\left(\frac{m+2}{2}\right) J_{m/2}\left(h\frac{r^2}{w_0^2}\right) \times \exp\left(-\frac{r^2}{w_0^2}\right)\exp(\pm im\theta), \tag{A.22}$$

$$\sigma^0 = \frac{\pi m! h^m w_0^2}{2(1+h^2)^{(m+1)/2}} \times F\left(\frac{m+1}{2}, \frac{m+1}{2}; m+1; \frac{h^2}{1+h^2}\right), \tag{A.23}$$

$$M^2 = \frac{m+1}{\sqrt{h^2+1}} \times \left[1 + \frac{h^2}{2} - \frac{h^4}{8}\frac{(m+1)}{(m+2)}\frac{F(\frac{m+3}{2}, \frac{m+3}{2}; m+3; -h^2)}{F(\frac{m+1}{2}, \frac{m+1}{2}; m+1; -h^2)}\right]. \tag{A.24}$$

(vii) Hypergeometric–Gaussian (HyGG) beams, $\mathrm{HyGG}_\mu^m(\mathbf{r}_t)$, [5]

$$P_0 = \lim_{p_0\to 0}\frac{\mathrm{i}k}{2}\left(\frac{1}{p_0} - \frac{1}{q_0}\right), \qquad S_0 = k\ \mathrm{Im}(1/q_0), \tag{A.25}$$

$$\mathrm{HyGG}_\mu^m(\mathbf{r}_t) = (P_0 r^2)^{m/2} {}_1F_1\left(-\frac{\mu}{2}, 1+m, P_0 r^2\right) \times \exp\left(\frac{\mathrm{i}kr^2}{2q_0}\right)\exp(\pm im\theta), \tag{A.26}$$

$$\sigma^0 = \frac{\pi m!}{k[\mathrm{Im}(1/q_0)]^{m+\mu+1}|q_0|^{m+\mu}}\frac{\Gamma(m+\mu+1)}{\Gamma(m+\mu/2+1)} \times \frac{(A+B/q_0)^{(\mu+m+2)/2}}{(2B)^{(\mu+m)/2}}. \tag{A.27}$$

Appendix B. F_2 Appell hypergeometric functions

In this appendix we include the definition and basic properties of the F_2 Appell hypergeometric functions. We refer the interested reader to [11, 12] for further information.

The F_2 Appell hypergeometric function of two variables (x, y) is defined by the series

$$F_2(a; b, b'; c, c'; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n}\frac{x^m y^n}{m!n!},$$
$$(|x| + |y| < 1; \quad \{c, c'\} \in \mathbb{C}\diagdown\mathbb{Z}_0^-; \quad \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}) \tag{B.1}$$

where $(a)_m$ denotes the Pochhammer symbol

$$(a)_m = a(a+1)(a+2)\cdots(a+m-1) = \frac{\Gamma(a+m)}{\Gamma(a)}, \tag{B.2}$$

with $\Gamma(x)$ being the Gamma function.

For zero and negative integer values of the parameters $b = -j = \{0, -1, -2, \ldots\}$ and $b' = -j' = \{0, -1, -2, \ldots\}$ the series (B.1) truncates as follows:

$$F_2(a; -j, -j'; c, c'; x, y) = \sum_{m=0}^{j}\sum_{n=0}^{j'}\frac{(a)_{m+n}}{(c)_m(c')_n}\frac{(-1)^{m+n}j!j'!}{(j-m)!(j'-n)!}\frac{x^m y^n}{m!n!}. \tag{B.3}$$

The Appell function F_2 has the following integral representation:

$$F_2(a; b, b'; c, c'; x, y) = \frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(b')\Gamma(c-b)\Gamma(c'-b')} \times \int_0^1\int_0^1 \frac{u^{b-1}v^{b'-1}(1-u)^{c-b-1}(1-v)^{c'-b'-1}}{(1-ux-vy)^a}\,\mathrm{d}u\,\mathrm{d}v, \tag{B.4}$$

where $|x| + |y| < 1$, $\mathrm{Re}\,c > \mathrm{Re}\,b > 0$, and $\mathrm{Re}\,c' > \mathrm{Re}\,b' > 0$.

The transformation formulae of the Appell function F_2 are

$$F_2(a; b, b'; c, c'; x, y) = F_2(a; b', b; c', c; y, x), \quad \text{(B.5)}$$

$$= (1-x)^{-a} F_2\left(a; c-b, b'; c, c'; \frac{x}{x-1}, \frac{y}{1-x}\right), \quad \text{(B.6)}$$

$$= (1-x-y)^{-a} F_2 \times \left(a; c-b, c'-b'; c, c'; \frac{x}{x+y-1}, \frac{y}{x+y-1}\right). \quad \text{(B.7)}$$

For some special cases, the Appell function F_2 can be expressed in terms of the conventional Gaussian hypergeometric function ${}_2F_1 = F$ as follows

$$F_2(a; b, b'; b, c'; x, y) = (1-x)^{-a} F\left(a, b'; c'; \frac{y}{1-x}\right) \quad \text{(B.8)}$$

$$F_2(a; b, b'; a, a; x, y) = (1-x)^{-b}(1-y)^{-b'} \times F\left(b, b'; a; \frac{xy}{(1-x)(1-y)}\right). \quad \text{(B.9)}$$

References

[1] Bandres M A and Gutiérrez-Vega J C 2008 Circular beams *Opt. Lett.* **33** 177–9

[2] Gori F, Guattari G and Padovani C 1987 Bessel–Gauss beams *Opt. Commun.* **64** 491–5

[3] Gutiérrez-Vega J C and Bandres M A 2005 Helmholtz–Gauss waves *J. Opt. Soc. Am.* A **22** 289–98

[4] Kotlyar V V, Skidanov R V, Khonina S N and Soifer V A 2007 Hypergeometric modes *Opt. Lett.* **32** 742–4

[5] Karimi E, Zito G, Piccirillo B, Marruci L and Santamato E 2007 Hypergeometric–Gaussian modes *Opt. Lett.* **32** 3053–5

[6] Kotlyar V V and Kovalev A A 2008 A family of hyper-geometric laser beams *J. Opt. Soc. Am.* A **25** 262–7

[7] Gutiérrez-Vega J C 2007 Fractionalization of optical beams: II. Elegant Laguerre–Gaussian modes *Opt. Express* **15** 6300–13

[8] Caron C F R and Potvliege R M 1999 Bessel-modulated Gaussian beams with quadratic radial dependence *Opt. Commun.* **15** 83–93

[9] Berry M V 2004 Optical vortices evolving from helicoidal integer and fractional phase steps *J. Opt. A: Pure Appl. Opt.* **6** 259–68

[10] Saad N and Hall R L 2003 Integrals containing confluent hypergeometric functions with applications to perturbed singular potentials *J. Phys. A: Math. Gen.* **36** 7771–88

[11] Gradshteyn I S and Ryzhik I M 2007 *Table of Integrals, Series, and Products* ed A Jeffrey and D Zwillinger (Amsterdam: Academic)

[12] Slater L J 1966 *Generalized Hypergeometric Functions* (New York: Cambridge University Press)

[13] Dodonov V V and Man'ko O V 2000 Universal invariants in quantum mechanics and physics of optical and particle beams *J. Russ. Laser Res.* **21** 438–64

[14] Belanger P A, Champagne Y and Pare C 1994 Beam propagation factor of diffracted laser beams *Opt. Commun.* **105** 233–42

[15] Saghafi S and Sheppard C J R 1998 The beam propagation factor for higher order Gaussian beams *Opt. Commun.* **153** 207–10

[16] Martínez-Herrero R, Piquero G and Mejías P M 1995 On the propagation of the kurtosis parameter of general beams *Opt. Commun.* **115** 225–32

[17] Porras M A, Borghi R and Santarsiero M 2001 Relationship between elegant Laguerre–Gauss and Bessel–Gauss beams *J. Opt. Soc. Am.* A **18** 177–84

[18] Lopez-Mago D, Bandres M A and Gutiérrez-Vega J C 2009 Propagation of Whittaker–Gaussian beams *Proc. SPIE* **7430** 7430–13