Circular beams

Miguel A. Bandres[1] and Julio C. Gutiérrez-Vega[2]

[1]*California Institute of Technology, Pasadena, California 91125, USA*
[2]*Photonics and Mathematical Optics Group, Tecnológico de Monterrey, Monterrey, México 64849*

Received November 8, 2007; revised December 4, 2007; accepted December 5, 2007;
posted December 10, 2007 (Doc. ID 89560); published January 11, 2008

A very general beam solution of the paraxial wave equation in circular cylindrical coordinates is presented. We call such a field a circular beam (CiB). The complex amplitude of the CiB is described by either the Whittaker functions or the confluent hypergeometric functions and is characterized by three parameters that are complex in the most general situation. The propagation through complex *ABCD* optical systems and the conditions for square integrability are studied in detail. Special cases of the CiB are the standard, elegant, and generalized Laguerre–Gauss beams; Bessel–Gauss beams; hypergeometric beams; hypergeometric–Gaussian beams; fractional-order elegant Laguerre–Gauss beams; quadratic Bessel–Gauss beams; and optical vortex beams. © 2008 Optical Society of America

OCIS codes: 260.1960, 070.2580, 070.2590, 350.5500.

In recent years, increasing interest has been paid to singular optics and beams carrying an intrinsic orbital angular momentum [1]. Paraxial beams exhibiting circular cylindrical symmetry, such as the standard, elegant, and generalized Laguerre–Gaussian beams (sLG, eLG, gLG) [2–4]; Bessel–Gauss (BG) beams [5,6]; the hypergeometric (HyG) beams [7], hypergeometric–Gauss beams (HyGG) [8,9]; fractional-order elegant LG beams (fr-eLG) [10]; BG beams with quadratic radial dependence (QBG) [11], and optical vortex beams (VB) [12], among others [13], have been extensively studied both theoretically and experimentally.

In this Letter, we introduce a very general beam solution of the paraxial wave equation (PWE) in circular cylindrical coordinates. These new solutions are called circular beams (CiBs) and are characterized by three parameters that are complex in the most general situation. The possibility of choosing arbitrary complex values for the beam parameters allows us to obtain novel and meaningful beam structures that, to our knowledge, have not yet been reported in the literature. For special values of the beam parameters, the CiBs reduce to the sLG, eLG, gLG, BG, HyG, HyGG, QBG, VB, and fr-eLG beams. The propagation of the CiB (and consequently of its known special cases) through complex *ABCD* optical systems and the conditions for square integrability of the CiB are also discussed in detail.

To obtain a general solution of the PWE

$$(\partial_x^2+\partial_y^2+2ik\partial_z)U(\mathbf{r},z)=0, \tag{1}$$

where $\mathbf{r}=(x,y)=(r\cos\theta,r\sin\theta)$ and k is the wavenumber, we propose the ansatz $U=Z(z)F(u,v)\mathrm{GB}(r,q)$, where $\boldsymbol{\rho}\equiv(u,v)\equiv(x,y)/\chi(z)$ are scaled Cartesian coordinates across the transverse plane; $\chi(z)$ is a z-dependent scaling factor to be determined; $\mathrm{GB}(r,q)$ is the fundamental Gaussian beam

$$\mathrm{GB}(r,q)=(1+z/q_0)^{-1}\exp[ikr^2/2q(z)], \tag{2}$$

with $q(z)=z+q_0$ being the known complex beam parameter for free-space propagation [2]; and q_0 is the beam parameter at the plane $z=0$. The parameter q_0 gives the radius of curvature R of the initial spherical phase front and the $1/e$ amplitude spot size w_0 of the initial Gaussian modulation through the relation $1/q_0=1/R+i2/kw_0^2$.

Substitution of the ansatz into Eq. (1) leads to the differential equations for $F(u,v)$, $\chi(z)$, and $Z(z)$:

$$[\bar{\nabla}^2-i\boldsymbol{\rho}\cdot\bar{\nabla}-\gamma-i]F=0, \tag{3a}$$

$$\partial_z(\chi^2/q^2)=1/kq^2, \tag{3b}$$

$$\partial_z Z/Z=(i\gamma-1)/2k\chi^2, \tag{3c}$$

where $\bar{\nabla}=(\partial_u,\partial_v)$ and γ is a separation constant.

Solving Eq. (3b) we obtain for the scaling parameter

$$1/\chi^2(z)=k(1/\tilde{q}-1/q), \tag{4}$$

where $\tilde{q}(z)=z+\tilde{q}_0$ and $\tilde{q}_0$ is an integration constant. The solution of Eq. (3c) gives $Z(z)=(\tilde{q}q_0/q\tilde{q}_0)^{i\gamma/2-1/2}$, such that $Z(0)=1$. Finally, assuming $F(\rho)=G(\rho)\exp(im\theta)\exp(i\rho^2/4)/\rho$, where $\rho=r/\chi(z)$ is the scaled radial coordinate and $m=0,\pm1,\pm2,\ldots$, and making the change of variables $\varrho=i\rho^2/2$, we get the differential equation for $G(\varrho)$:

$$\left[\partial_{\varrho\varrho}-\frac{1}{4}+\frac{i\gamma/2}{\varrho}+\frac{1/4-(m/2)^2}{\varrho^2}\right]G(\varrho)=0. \tag{5}$$

This equation is identified as the canonical form of the Whittaker differential equation [14] whose solutions are given by $M_{i\gamma/2,m/2}(\varrho)$ Whittaker functions of argument ϱ and parameters γ and m.

By collecting the partial results and rearranging terms we obtain

$$\mathrm{CiB}_\gamma^m(\mathbf{r};q,\tilde{q})=\left(\frac{\tilde{q}/\tilde{q}_0}{q/q_0}\right)^{i\gamma/2}\left(\frac{ir^2}{2\chi^2}\right)^{-1/2}M_{i\gamma/2,m/2}\left(\frac{ir^2}{2\chi^2}\right)\times[\mathrm{GB}(r,q)\mathrm{GB}(r,\tilde{q})]^{1/2}\exp(im\theta). \tag{6a}$$

The known relations between the Whittaker functions $M_{\kappa,\mu}$ and the confluent hypergeometric function ${}_1F_1(a,b;x)$ (Chap. 13 of [14]) allow us to express the wave CiB_γ^m in the following equivalent form:

$$\mathrm{CiB}_{\gamma}^{m}(\mathbf{r};q,\tilde{q}) = \left(\frac{\tilde{q}/\tilde{q}_0}{q/q_0}\right)^{i\gamma/2-1/2}\left(\frac{ir^2}{2\chi^2}\right)^{m/2} \times {}_1F_1\left(\frac{1+m-i\gamma}{2},1+m,\frac{ir^2}{2\chi^2}\right) \times \mathrm{GB}(r,q)\exp(im\theta). \quad (6b)$$

Equations (6) are exact solutions of the PWE in free space [Eq. (1)] and represent the first important result of this Letter. At any transverse z plane, the structure of the CiBs is characterized by an arbitrarily complex parameter γ and two independent complex beam parameters $q(z)$ and $\tilde{q}(z)$. It is important that $\tilde{q}$ is a new independent complex parameter of the beam solutions. Under free-space propagation through a distance z, the field parameters transform according to the translation property $(\gamma,q_0,\tilde{q}_0)\mapsto(\gamma,q_0+z,\tilde{q}_0+z)$, retaining the functional form of the field. An important symmetry property of the $\mathrm{CiB}_{\gamma}^{m}$ [Eqs. (6)] is that they are invariant under the transformation $(\gamma,q,\tilde{q})\leftrightarrow(-\gamma,\tilde{q},q)$.

Equations (6) describe the propagation of the CiBs in free space. We will now study the propagation of the CiBs through more general types of paraxial optical systems characterized by complex *ABCD* matrices. Consider that a CiB with parameters $(\gamma,q_0,\tilde{q}_0)$ at the input plane (r_0,z_0) of a paraxial system has the general form

$$\Psi_0(\mathbf{r}_0;q_0,\tilde{q}_0) = \left(\frac{ir_0^2}{2\chi_0^2}\right)^{m/2}\exp(im\theta)\exp\left(\frac{ikr_0^2}{2q_0}\right) \times {}_1F_1\left(\frac{1+m-i\gamma}{2},1+m,\frac{ir_0^2}{2\chi_0^2}\right). \quad (7)$$

The field at the output plane $(\mathbf{r}_1,z_1)$ of the *ABCD* system is determined by solving the Huygens diffraction integral in cylindrical coordinates [2]

$$\Psi_1(r_1;q_1,\tilde{q}_1) = \frac{(-i)^{m+1}k}{B}\int_0^{\infty}\mathrm{d}r_0\Psi_0(\mathbf{r}_0;q_0,\tilde{q}_0) \times r_0J_m\left(\frac{kr_0r_1}{B}\right)\exp\left[\frac{ik}{2B}(Ar_0^2+Dr_1^2)\right], \quad (8)$$

where J_m is the Bessel function of mth order. The integration yields

$$\Psi_1(\mathbf{r}_1;q_1,\tilde{q}_1) = \left(\frac{A+B/\tilde{q}_0}{A+B/q_0}\right)^{i\gamma/2-1/2}\left(\frac{ir_1^2}{2\chi_1^2}\right)^{m/2} \times {}_1F_1\left(\frac{m+1-i\gamma}{2},m+1;\frac{ir_1^2}{2\chi_1^2}\right) \times \mathrm{GB}(r_1,q_1)\exp(im\theta), \quad (9)$$

where

$$\mathrm{GB}(r_1,q_1) = (A+B/q_0)^{-1}\exp(ikr_1^2/2q_1) \quad (10)$$

is the output field of a Gaussian beam with input parameter q_0 traveling axially through the *ABCD* system; the transformation laws for the beam parameters from input plane z_0 to output plane z_1 are

$$q_1 = \frac{Aq_0+B}{Cq_0+D}, \qquad \tilde{q}_1 = \frac{A\tilde{q}_0+B}{C\tilde{q}_0+D}. \quad (11)$$

Equation (9) is the second important result of this Letter. It allows a CiB to be traced through an arbitrary real or complex *ABCD* optical system. The output function for a CiB with input parameters $(\gamma,q_0,\tilde{q}_0)$ propagating through a paraxial system is found to have the same functional form as the input function but with new values for the parameters $(\gamma,q_1,\tilde{q}_1)$. As expected, Eq. (9) reduces to Eq. (6a) for the case of free-space propagation, i.e., when $ABCD=[1,z;0,1]$.

At both the input and output planes, the scaling parameter χ is determined from the complex beam parameters q and $\tilde{q}$ according to Eq. (4). By using Eqs. (11) it is easy to show that χ_1^2 may also be written in the equivalent forms

$$\chi_1^2 = \chi_0^2(A+B/q_0)(A+B/\tilde{q}_0), \quad (12)$$

$$\chi_1^2 = (A+B/q_0)^2\{\chi_0^2+[(B/k)/(A+B/q_0)]\}. \quad (13)$$

Equation (13) can be recognized as the transformation law for the "complex spot size" of the generalized Hermite–Gaussian beams studied by Siegman (p. 798 of [2]). This new parameterization that we have introduced to describe the CiB in terms of the two independent complex beams parameters q and $\tilde{q}$ is more convenient for the following reasons: first, the transformation law for $\tilde{q}$ takes exactly the same and well-known bilinear relation as that for the complex beam parameter q [Eqs. (11)]. Second, the invariance of the CiBs under the transformation $(\gamma,q,\tilde{q})\leftrightarrow(-\gamma,\tilde{q},q)$ is clearly evident. Finally, the conditions for square integrability of the CiB are expressed in a simpler and symmetrical form.

Particular values of the beam parameters lead to known solutions of the PWE that have been well studied in paraxial optics. Table 1 shows a list of these special cases and the corresponding values of $(\gamma,q_0,\tilde{q}_0)$ at the initial plane $z_0=0$. From Table 1 we see that for some subsets of γ and fixed q and $\tilde{q}$ the CiBs are (bi)orthogonal and complete. Notice that the relations between the known special cases, which have not been clear in the past, now become evident due to our parameterization of the CiBs. For example, in [15] an asymptotic relation between the eLG of high radial order n and BG beams is studied; from Table 1 we indeed conclude that not only the eLG beams but in general the gLG beams with high radial order n tend to BG beams with $k_t^2=i2nk(1/\tilde{q}_0-1/q_0)$ and complex beam parameter $1/q_{\mathrm{BG}}=1/2q_0+1/2\tilde{q}_0$.

Table 1. Special Cases of $\mathrm{CiB}_{\gamma}^{m}(\mathbf{r};q_0,\tilde{q}_0)$

Special Cases	γ	q_0	$\tilde{q}_0$	Comment
sLG_n^m	$-i(2n+m+1)$	q_0	q_0^*	$n=0,1,2,\ldots$
eLG_n^m	$-i(2n+m+1)$	q_0	∞	$n=0,1,2,\ldots$
gLG_n^m	$-i(2n+m+1)$	q_0	$\tilde{q}_0$	$n=0,1,2,\ldots$
HyGG_p^m	$-i(p+m+1)$	q_0	0	$p\in\mathbb{R}^a$
fr-eLG_η^m	$-i(2\eta+m+1)$	q_0	∞	$\eta\in\mathbb{R}$
$\mathrm{BG}_{k_t}^m$	$-i(2n+m+1)$	q_0	$(q_0^{-1}-ik_t^2/2nk)^{-1}$	$\lim_{n\to\infty}\Psi_1\propto J_m(k_t r)\exp(ikr^2/2\tilde{q}_0-k_t^2r^2/8n)$
VB^m	$-i$	∞	0	$\mathrm{VB}^m(r,\theta,0)\propto\exp(im\theta)$
HyG_γ^m	γ	∞	0	$\gamma\in\mathbb{R}$
WGB_γ^m	γ	$-\tilde{q}_0^*$	$k(i2/w_0^2+1/2W_0^2)^{-1}$	$\{w_0,W_0,\gamma\}\in\mathbb{R}$
QBG^m	0	$-\tilde{q}_0^*$	$k(i2/w_0^2+1/2W_0^2)^{-1}$	$\{w_0,W_0\}\in\mathbb{R}$

[a] For $p\in\mathbb{C}$ see [9].

The combination of values in the penultimate row of Table 1 produces a new structure that we call a Whittaker–Gaussian beam (WGB), whose field at the plane $z=0$ reads as

$$\mathrm{WGB}_\gamma^m(\mathbf{r})=\left[\left(\frac{ir^2}{2W_0^2}\right)^{-1/2}M_{i\gamma/2,m/2}\left(\frac{ir^2}{2W_0^2}\right)\right]\times\exp(-r^2/w_0^2)\exp(im\theta), \tag{14}$$

where $\gamma\in\mathbb{R}$ and w_0 is the width of the Gaussian envelope. The expression in brackets has constant phase, and its scale is controlled by $W_0\in\mathbb{R}$. The beams WGB_γ^m carry finite power and form a biorthogonal set of solutions of the PWE. As shown in the last row of Table 1, the special case of the WGBs when $\gamma=0$ corresponds to the QBG beams introduced in [11]. To our best knowledge, the WGB has not yet been reported in the optics literature, and its physical properties are currently under study by us.

We remark that the general expression Eq. (9) can be applied straightforwardly to propagate all special cases reported in Table 1 through *ABCD* optical systems, including the gLG, fr-eLG, HyG, HyGG, QBG, and VB beams whose propagation through *ABCD* systems has not been analyzed previously to our knowledge.

From a physical point of view, it is important to identify the range of values of $(\gamma,q,\tilde{q})$ for which the CiBs transport finite power, i.e., for which are square integrable across the whole transverse plane. First, the case $\gamma=-i(2n+m+1)$ with $n=0,1,2,\ldots$ leads to sLG, eLG, gLG, BG, and HyGG beams with p even for which the square integrability is ensured by setting $\mathrm{Im}(1/q)>0$.

For arbitrary $\gamma\neq -i(2n+m+1)$, consider an integrability plane whose axes are associated with $\mathrm{Im}(1/q)$ and $\mathrm{Im}(1/\tilde{q})$. Each point on the plane is associated with the pair of values $[\mathrm{Im}(1/q),\mathrm{Im}(1/\tilde{q})]$ that the beam acquires at a given transverse z plane. The beam (a) is square integrable if the point falls within the first quadrant of the integrability plane, and (b) diverges as $|x|\to\infty$ if the point falls within the second, third, or fourth quadrants. If the point falls on the positive $\mathrm{Im}(1/q)$ axis then the square integrability depends on γ as follows: the beam (i) is square integrable if $\mathrm{Im}\ \gamma<0$; (ii) tends to zero as $|x|\to\infty$ but it is not square integrable if $\mathrm{Im}\ \gamma\in[0,1/2)$; (iii) tends to a constant nonzero value as $|x|\to\infty$ if $\mathrm{Im}\ \gamma=1/2$; (iv) diverges if $\mathrm{Im}\ \gamma>1/2$. If the point falls on the positive $\mathrm{Im}(1/\tilde{q})$ axis then we have the same cases as in the $\mathrm{Im}(1/\tilde{q})$ axis but with γ replaced by $-\gamma$. Finally, if $\mathrm{Im}(1/q)=\mathrm{Im}(1/\tilde{q})=0$, we have again the cases (ii) if $\mathrm{Im}\ \gamma<1/2$, (iii) if $\mathrm{Im}\ \gamma=1/2$, and (iv) if $\mathrm{Im}\ \gamma>1/2$.

We acknowledge support from Consejo Nacional de Ciencia y Tecnología (grant 42808), the Tecnológico de Monterrey (grant CAT-007), and the Secretaría de Educación Pública de México.

References

1. M. S. Soskin and M. Vasnetsov, Prog. Opt. **42**, 219 (2001).
2. A. E. Siegman, *Lasers* (University Science, 1986).
3. A. Wunsche, J. Opt. Soc. Am. A **6**, 1320 (1989).
4. M. A. Bandres and J. C. Gutiérrez-Vega, Proc. SPIE **6290**, 6290-0S (2006).
5. F. Gori, G. Guattari, and C. Padovani, Opt. Commun. **64**, 491 (1987).
6. J. C. Gutiérrez-Vega and M. A. Bandres, J. Opt. Soc. Am. A **22**, 289 (2005).
7. V. V. Kotlyar, R. V. Skidanov, S. N. Khonina, and V. A. Soifer, Opt. Lett. **32**, 742 (2007).
8. E. Karimi, G. Zito, B. Piccirillo, L. Marruci, and E. Santamato, Opt. Lett. **32**, 3053 (2007).
9. V. V. Kotlyar and A. A. Kovalev, J. Opt. Soc. Am. A **25**, 262 (2008).
10. J. C. Gutiérrez-Vega, Opt. Express **15**, 6300 (2007).
11. C. F. R. Caron and R. M. Potvliege, Opt. Commun. **164**, 83 (1999).
12. M. V. Berry, J. Opt. A, Pure Appl. Opt. **6**, 259 (2004).
13. C. P. Boyer, E. G. Kalnins, and W. Miller, J. Math. Phys. **16**, 499 (1975).
14. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, 1964).
15. M. A. Porras, R. Borghi, and M. Santarsiero, J. Opt. Soc. Am. A **18**, 177 (2001).