Cartesian beams

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A new and very general beam solution of the paraxial wave equation in Cartesian coordinates is presented. We call such a field a Cartesian beam. The complex amplitude of the Cartesian beams is described by either the parabolic cylinder functions or the confluent hypergeometric functions, and the beams are characterized by three parameters that are complex in the most general situation. The propagation through complex *ABCD* optical systems and the conditions for square integrability are studied in detail. Applying the general expression of the Cartesian beams, we also derive two new and meaningful beam structures that, to our knowledge, have not yet been reported in the literature. Special cases of the Cartesian beams are the standard, elegant, and generalized Hermite–Gauss beams, the cosine-Gauss beams, the Lorentz beams, and the fractional order beams. © 2007 Optical Society of America
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The role of the paraxial approximation in the propagation of optical fields has been well established since long ago [1]. Many closed-form solutions of the paraxial wave equation (PWE) have been widely studied from theoretical and experimental points of view, for example, the families of Hermite–Gaussian (HG), Laguerre–Gaussian [1], and Ince–Gaussian beams [2], the Helmholtz–Gauss and Laplace–Gauss beams [3], and Hermite sinusoidal Gaussian beams, among many others [4].

In this Letter, we introduce a new and very general beam solution of the PWE in Cartesian coordinates. These new solutions are termed Cartesian beams and are characterized by three parameters that are complex in the most general situation. The possibility of choosing arbitrary complex values for the beam parameters allows us to obtain novel and meaningful beam structures, that, to our knowledge, have not yet been reported in the literature. For special values of the beam parameters, the Cartesian beams reduce to the standard, elegant, and generalized HG beams [1,5,6], the cosine-Gauss or cosh-Gauss beams [3], the Lorentz beams [7], and the fractional order beams [8]. The propagation of the Cartesian beams (and consequently of their known special cases) through complex *ABCD* optical systems is also studied, considering the possibility of having an initial transverse displacement and tilt. The conditions for square integrability of the Cartesian beams are discussed in detail.

Before beginning the analysis, we first note that since the fields can be split into products of solutions in the x and y directions, without loss of generality, we consider solutions of the PWE in only one Cartesian coordinate. To obtain a general solution of the one-dimensional PWE,

$$(\partial_x^2 + 2ik\partial_z)U(x,z) = 0, \tag{1}$$

where k is the wavenumber, we propose the ansatz $U(x,z)=Z(z)F(u)\mathrm{GB}(x,q)$, where $u \equiv x/\chi(z)$ is a scaled coordinate along the x direction, $\chi(z)$ is the z-dependent scaling factor to be determined, $\mathrm{GB}(x,q)$ is the one-dimensional fundamental Gaussian beam

$$\mathrm{GB}(x,q) = (1 + z/q_0)^{-1/2} \exp[ikx^2/2q(z)], \tag{2}$$

$q(z)=z+q_0$ is the known complex beam parameter for free-space propagation [1], and q_0 is the beam parameter at the plane $z=0$.

Substitution of the ansatz into Eq. (1) leads to the differential equations for $F(u)$, $\chi(z)$, and $Z(z)$:

$$[\partial_u^2 - iu\partial_u - \alpha - i/2]F = 0, \tag{3a}$$

$$\partial_z(\chi^2/q^2) = 1/kq^2, \tag{3b}$$

$$\partial_z Z/Z = (i\alpha - 1/2)/2k\chi^2, \tag{3c}$$

where α is a separation constant.

Solving Eq. (3b) we obtain for the scaling parameter

$$\chi(z) = [k(1/\tilde{q} - 1/q)]^{-1/2}, \tag{4}$$

where $\tilde{q}(z)=z+\tilde{q}_0$, and $\tilde{q}_0$ is an integration constant. The solution of Eq. (3c) gives $Z(z)=(\tilde{q}q_0/q\tilde{q}_0)^{i\alpha/2-1/4}$, such that $Z(0)=1$. Finally, assuming $F(u)=G(u)\exp(iu^2/4)$, we get the differential equation $(d_u^2+u^2/4-\alpha)G(u)=0$ for $G(u)$. This equation is easily identified as the canonical form of the parabolic cylinder differential equation (Sect. 19.16 of [9]), whose independent solutions are given by the even $\mathrm{P}_\alpha^{\mathrm{e}}(u)$ and odd $\mathrm{P}_\alpha^{\mathrm{o}}(u)$ parabolic cylinder functions of argument u and parameter α. We adopt the $\mathrm{P}_\alpha^{\mathrm{e}}$ and $\mathrm{P}_\alpha^{\mathrm{o}}$ functions, instead of the more familiar functions D_ν (Sect. 19.3 of [9]), because the former make explicit the symmetry and antisymmetry of the two independent families of beam solutions.

By collecting the partial results and rearranging terms we obtain

$$U_\alpha^{\mathrm{e,o}}(x;q,\tilde{q}) = \left(\frac{\tilde{q}/\tilde{q}_0}{q/q_0}\right)^{i\alpha/2} \mathrm{P}_\alpha^{\mathrm{e,o}}\left(\frac{x}{\chi(z)}\right) \times [\mathrm{GB}(x,q)\mathrm{GB}(x,\tilde{q})]^{1/2}. \tag{5a}$$

The known relations between the parabolic cylinder functions $\mathrm{P}_\alpha^{\mathrm{e,o}}$ and the confluent hypergeometric function ${}_1F_1(a,b;x)$ (Chap. 19 of [9]) allow us to express the wave $U_\alpha^{\mathrm{e,o}}$ in the following equivalent form:

$$U_\alpha^{\mathrm{e,o}}(x;q,\tilde{q}) = (\tilde{q}q_0/q\tilde{q}_0)^{i\alpha/2-1/4}\mathrm{GB}(x,q) \times \left(\frac{x}{\chi}\right)^{0,1} {}_1F_1\left(\frac{2\mp1}{4} - \frac{i\alpha}{2}, \frac{2\mp1}{2}; \frac{ix^2}{2\chi^2}\right). \tag{5b}$$

Equations (5) are exact solutions of the PWE in free space [Eq. (1)] and represent the first important result of this Letter. At any transverse z plane, the structure of the Cartesian beams is characterized by an arbitrarily complex parameter α and two independent complex beam parameters $q(z)$ and $\tilde{q}(z)$. It is important to emphasize that $\tilde{q}$ is a new independent complex parameter of the beam solutions. Under free-space propagation through a distance z, the field parameters transform according to the translation property $(\alpha,q_0,\tilde{q}_0)\mapsto(\alpha,q_0+z,\tilde{q}_0+z)$, retaining the functional form of the wave. An important symmetry property of the Cartesian beams $U_\alpha^{\mathrm{e,o}}$ [Eqs. (5)] is that they are invariant under the transformation $(\alpha,q,\tilde{q})\leftrightarrow(-\alpha,\tilde{q},q)$. Two-dimensional solutions of the PWE in Cartesian coordinates can be readily constructed with products of one-dimensional solutions of the form (5), that is, $U_{\alpha,\beta}(\mathbf{r}) = U_\alpha^{e,o}(x;q_x,\tilde{q}_x)U_\beta^{e,o}(y;q_y,\tilde{q}_y)$, where any combination of parities is possible.

Equations (5) describe the propagation of the Cartesian beams $U_\alpha^{e,o}$ in free space. We will now study the propagation of the Cartesian beams through more general types of paraxial optical systems characterized by complex $ABCD$ matrices. Consider that a Cartesian beam with parameters $(\alpha,q_0,\tilde{q}_0)$ at the input plane (x_0,z_0) of a paraxial system has the general form

$$\Psi_0^{\mathrm{e,o}}(x_0;q_0,\tilde{q}_0) = \exp\left(\frac{ikx^2}{2q_0}\right)\exp(iS_0x_0)\left(\frac{x_0+\delta_0}{\chi_0}\right)^{0,1} \times {}_1F_1\left(\frac{2\mp1}{4} - \frac{i\alpha}{2}, \frac{2\mp1}{2}; \frac{i(x_0+\delta_0)^2}{2\chi_0^2}\right), \tag{6}$$

where the new parameters δ_0 and S_0 are complex quantities in general and allow for the possibility that the input field $\Psi_0^{\mathrm{e,o}}$ has an initial transverse displacement and tilt, respectively.

The field at the output plane (x_1,z_1) of the $ABCD$ system is determined by solving the Huygens diffraction integral [1]

$$\Psi_1^{\mathrm{e,o}}(x_1;q_1,\tilde{q}_1) = \sqrt{k/i2\pi B}\int_{-\infty}^{\infty}\Psi_0^{\mathrm{e,o}}(x_0;q_0,\tilde{q}_0) \times \exp[ik(Ax_0^2 - 2x_0x_1 + Dx_1^2)/2B]\mathrm{d}x_0. \tag{7}$$

The integration yields

$$\Psi_1^{\mathrm{e,o}}(x_1;q_1,\tilde{q}_1) = \left(\frac{A+B/\tilde{q}_0}{A+B/q_0}\right)^{i\alpha/2-1/4}\exp\left(\frac{BS_0S_1}{i2k}\right) \times \left(\frac{x_1+\delta_1}{\chi_1}\right)^{0,1} {}_1F_1\left(\frac{2\mp1}{4} - \frac{i\alpha}{2}, \frac{2\mp1}{2}; \frac{i(x_1+\delta_1)^2}{2\chi_1^2}\right) \times \mathrm{GB}(x_1,q_1)\exp(iS_1x_1), \tag{8}$$

where

$$\mathrm{GB}(x_1,q_1) = (A+B/q_0)^{-1/2}\exp(ikx_1^2/2q_1) \tag{9}$$

is the output field of a Gaussian beam with input parameter q_0 traveling axially through the $ABCD$ system and the transformation laws for the beam parameters from the input plane z_0 to the output plane z_1 are

$$q_1 = \frac{Aq_0+B}{Cq_0+D}, \quad \tilde{q}_1 = \frac{A\tilde{q}_0+B}{C\tilde{q}_0+D}, \tag{10}$$

$$\delta_1 = \delta_0(A+B/q_0) - BS_0/k, \tag{11}$$

$$S_1 = S_0/(A+B/q_0). \tag{12}$$

Equation (8) is the second important result of this Letter. It allows a Cartesian beam to be traced through an arbitrary real or complex $ABCD$ optical system including even the possibility of having a transverse displacement and tilt at the input plane. The output function for a Cartesian beam with input parameters $(\alpha,q_0,\tilde{q}_0,\delta_0,S_0)$ propagating through a paraxial system is found to have the same functional form as the input function but with new values for the parameters $(\alpha,q_1,\tilde{q}_1,\delta_1,S_1)$. As expected, Eq. (8) reduces to Eq. (5) for the case of free-space propagation, i.e., when $ABCD=[1,z;0,1]$.

At both the input and output planes, the scaling parameter χ is determined from the complex beam parameters q and $\tilde{q}$ according to Eq. (4). Using Eqs. (10) it is easy to show that χ_1^2 may also be written in the equivalent forms

$$\chi_1^2 = \chi_0^2(A+B/q_0)(A+B/\tilde{q}_0), \tag{13}$$

$$\chi_1^2 = (A+B/q_0)^2\left[\chi_0^2 + \frac{B/k}{(A+B/q_0)}\right]. \tag{14}$$

Equation (14) can be recognized as the transformation law for the "complex spot size" of the generalized HG beams studied by Siegman (p. 798 of [1]). This new parameterization that we have introduced to describe the Cartesian beams in terms of the two independent complex beams parameters q and $\tilde{q}$ is more convenient for the following reasons: first, the transformation law for $\tilde{q}$ takes exactly the same and well-known bilinear relation as that for the complex beam parameter q [Eqs. (10)]. Second, the invariance of the Cartesian beams under the transformation $(\alpha,q,\tilde{q})\leftrightarrow(-\alpha,\tilde{q},q)$ is clearly evident. Finally, the conditions for square integrability of the Cartesian beams are expressed in a simpler and symmetrical form.

Particular values of the beam parameters lead to known solutions of the PWE that have been well studied in paraxial optics. Table 1 shows a list of these special cases and the corresponding values of $(\alpha, q_0, \tilde{q}_0)$ at the initial plane z_0. It is interesting to note that the already known special cases correspond to purely imaginary values of the parameter α in all cases. The possibility of choosing arbitrary complex values for the parameters $(\alpha, q, \tilde{q})$ allows us to obtain new meaningful beam structures that, to our knowledge, have not yet been reported in the optics literature.

For example, the combination of values in the eighth row of Table 1 produces a new solution $\Phi_\alpha^{e,o}$ whose free-space propagation for $z_S=0$ is given by

$$\Phi_\alpha^{e,o}(x,z) = \frac{\exp(ikx^2/4z)}{z^{1/4+i\alpha/2}} \mathrm{P}_\alpha^{e,o}\left(\frac{x}{\sqrt{z/k}}\right). \tag{15}$$

When $\alpha \in \mathbb{R}$ the solutions described by Eq. (15) form a complete and orthogonal basis for the space of square integrable functions on the real line [4]. For $\alpha = -i/2$, $\Phi_\alpha^{e,o}(x,z)$ reduces to the known paraxial spherical wave (p. 637 of [1]), i.e., $\exp[ikx^2/2z]/\sqrt{z}$.

A second new solution of the one-dimensional PWE is given in the ninth row of Table 1, for which $(\tilde{q}_0)^{-1} = -(q_0^*)^{-1} = i2/kw_0^2 + 1/2kW_0^2$ and $\{w_0, W_0, \alpha\} \in \mathbb{R}$. The new solutions $\Theta_\alpha^{e,o}$ are expressed in terms of parabolic cylinder functions apodized by a Gaussian envelope, e.g., at $z=0$ we have

$$\Theta_\alpha^{e,o}(x,0) = \mathrm{P}_\alpha^{e,o}(x/W_0)\exp(-x^2/w_0^2). \tag{16}$$

The beams $\Theta_\alpha^{e,o}$ carry finite power and form a biorthogonal set of solutions of the PWE. To our best knowledge, the paraxial wave solutions included in the rows 8 and 9 of Table 1 have not yet been reported in the optics literature, and their physical properties are currently under study by the authors.

Table 1. Special Cases of $U_\alpha^{e,o}(x;q_0,\tilde{q}_0)$

Known Special Cases	α	q_0	$\tilde{q}_0$
Gaussian beam[a]	$-i/2$	q_0	–
Standard HG_n[b]	$-i(n+1/2)$	q_0	q_0^*
Elegant HG_n[b]	$-i(n+1/2)$	q_0	∞
Generalized HG_n[b]	$-i(n+1/2)$	q_0	$\tilde{q}_0$
Cosine-Gauss beams[a,c]	$-i(n+1/2)$	q_0	c
Lorentz beams[d]	$i/2$	∞	$\tilde{q}_0$
Fractional beams[e]	$-i(p+1/2)$	q_0	∞
New Special Cases			
$\Phi_\alpha^{e,o}(x)$ [Eq. (15)]	α	$-z_S$	∞
$\Theta_\alpha^{e,o}(x)$ [Eq. (16)]	α	q_0	$-q_0^*$

[a]Obtained with U_α^e.
[b]$\mathrm{HG}_{n=0,2,4,\ldots}$ with U_α^e, and $\mathrm{HG}_{n=1,3,5,\ldots}$ with U_α^o.
[c]$\tilde{q}_0^{-1} = q_0^{-1} - ik_t^2/2nk$. In the limit when $n\to\infty$, then U_α^e tends to the cosine-Gauss beam, i.e., $\cos(k_t x)\exp(ikx^2/2\tilde{q} - kt^2x^2/8n)$.
[d]Lorentz beam Eq. (16) in [7] is obtained with the superposition $\sqrt{\pi}U_\alpha^e(x\pm iw) - \sqrt{-i2}U_\alpha^o(x\pm iw)$.
[e]p-order derivative fractional beams [8] are obtained with $\cos(\pi p/2)\Gamma(1+p/2)U_\alpha^e(x) - \sqrt{1/2}i\sin(\pi p/2)p\Gamma(p/2)U_\alpha^o(x)$.

We remark that the general expression Eq. (8) can be applied straightforwardly to propagate all special cases reported in Table 1 through *ABCD* optical systems, including the Lorentz [7] and fractional beams [8], whose propagation through *ABCD* systems has not been analyzed previously.

From a physical point of view, it is important to identify the range of values of $(\alpha, q, \tilde{q})$ for which the Cartesian beams transport finite power, i.e., for which the beams are square integrable. First, the case $\alpha = -i(n+1/2)$ with $n - \delta_{\text{parity},o} = 0, 2, 4, \ldots$ leads to standard, elegant, and generalized HG beams for which the square integrability is ensured by setting $\mathrm{Im}(1/q) > 0$.

For arbitrary $\alpha \neq -i(n+1/2)$, consider an integrability plane whose axes are associated with $\mathrm{Im}(1/q)$ and $\mathrm{Im}(1/\tilde{q})$ and with azimuthal angle $\phi = \arctan[\mathrm{Im}(1/\tilde{q})/\mathrm{Im}(1/q)] \in [0, 2\pi)$. Each point on the plane is associated with the pair of values $[\mathrm{Im}(1/q), \mathrm{Im}(1/\tilde{q})]$ that the beam acquires at a given transverse z plane. The beam (a) is square integrable if the point falls within the first quadrant [i.e., $\phi \in (0, \pi/2)$] of the integrability plane and (b) diverges as $|x|\to\infty$ if the point falls within the second, third, or fourth quadrants [i.e., $\phi \in (\pi/2, 2\pi)$]. If the point falls on the positive $\mathrm{Im}(1/q)$-axis (i.e., $\phi=0$) then the square integrability depends on α as follows: the beam (i) is square integrable if $\mathrm{Im}\,\alpha < 0$, (ii) tends to zero as $|x|\to\infty$ but is not square integrable if $\mathrm{Im}\,\alpha \in [0, 1/2)$, (iii) tends to a constant nonzero value as $|x|\to\infty$ if $\mathrm{Im}\,\alpha = 1/2$, and (iv) diverges if $\mathrm{Im}\,\alpha > 1/2$. If the point falls on the positive $\mathrm{Im}(1/\tilde{q})$-axis (i.e., $\phi = \pi/2$), then we have the same cases as in $\phi=0$ but with α replaced by $-\alpha$. Finally, if $\mathrm{Im}(1/q) = \mathrm{Im}(1/\tilde{q}) = 0$, we have again the cases (ii) if $\mathrm{Im}\,\alpha < 1/2$, (iii) if $\mathrm{Im}\,\alpha = 1/2$, and (iv) if $\mathrm{Im}\,\alpha > 1/2$.

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