Self-trapped modes in highly nonlocal nonlinear media

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Exact analytical solutions describing spatially localized self-trapped modes in highly nonlocal nonlinear media are presented. We formulate the model in a coordinate-free form and show that it allows us to obtain useful closed-form expressions for a large variety of mode structures and their interactions in a highly nonlocal nonlinear medium.

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I. INTRODUCTION

In recent times there are strongly increasing activities on the nonlinear light propagation in nonlocal media, both from an experimental and theoretical side [1–13]. Among the new physical effects attributed to nonlocality are, for example, the suppression of beam collapse [14], and the stabilization of nonlinear structures that are known to be unstable in realistic local media [15], e.g., dipole solitons [16], vortex solitons [17,18], Laguerre and Hermite cluster solitons [19], and azimuthons, that are intermediate structures between scalar multipole solitons and pure vortex solitons [20,21].

Several approaches have been available to model propagation in nonlocal nonlinear media. In particular, the highly nonlocal limit occurs when the characteristic nonlocal response is much larger than the beam size. Snyder and Mitchell [1] introduced a simple theoretical model to describe the propagation of solitons in a highly nonlocal nonlinear (HNN) medium. In a beginning, the model was not fully explored, mainly because of the apparent lack of a physical HNN medium able to support such solitons. Nevertheless, in recent years, the applicability of this model has witnessed a revival of research interest [2] after the observation of optical solitons in HNN nematic liquid displays [3,4], in photorefractive media [5], and in lead glasses exhibiting self-focusing thermal nonlinearity [6,7]. These experimental observations provide independent proofs of the highly nonlocal dynamics of some physical systems and turns into reality the possibility of observing the accessible solitons predicted in Ref. [1].

In this paper we present exact analytical solutions describing spatially localized self-trapped modes in HNN media, the so-called Helmholtz-Gauss (HzG) modes. We follow a coordinate-free approach rather than proposing solutions for the nonlocal nonlinear Schrödinger equation in a particular coordinate system. This formulation captures the essence of the mode propagation and is therefore more robust. The choice of coordinate system then enters into the model in a later stage for describing several special cases. The model provides deep insight into the wave dynamics in HNN media and allows us to obtain useful closed-form expressions for a large variety of self-trapped modes and their interactions. The solutions also will be useful in different contexts both for the case of strongly nonlocal media, which can be realized in experiments, and in the context of beam propagation in linear graded index media. Finally, we remark that the solutions presented in this work correspond effectively to self-trapped modes in HNN media and cannot be called formally solitons in the sense that the term soliton corresponds to a nonlinear wave whose transverse structure is shape invariant under propagation.

II. SELF-TRAPPED HELMHOLTZ-GAUSS MODES

We begin by considering the amplitude $E(\mathbf{r},z)$ of a beam propagating in a nonlocal nonlinear medium along the z axis of a coordinate system $(\mathbf{r},z)$, where $\mathbf{r}=(x,y)=(r,\theta)$ denotes the transverse coordinates. The paraxial propagation of the beam is governed by the nonlocal nonlinear Schrödinger equation [1,2,15]

$$2ik\frac{\partial E}{\partial z}+\nabla_\perp^2 E+\frac{2k^2}{n_0}E\int R(\mathbf{r}-\mathbf{r}')|E(\mathbf{r}',z)|^2dx'dy'=0, \tag{1}$$

where $\nabla_\perp^2$ is the transverse Laplacian, k is the wave number in the linear medium, n_0 is the linear part of the refractive index, and $R(\mathbf{r}-\mathbf{r}')$ is the normalized radially symmetric spatial nonlocal response function of the medium. For HNN media, the function $R(\mathbf{r}-\mathbf{r}')$ is much wider than the beam width, then it can be expanded using Taylor series with respect to $\mathbf{r}'$ about $\mathbf{r}'=\mathbf{r}$. In the limit of strongly nonlocal response only the dominant term of the series is kept, thus Eq. (1) reduces to

$$\left(2ik\frac{\partial}{\partial z}+\nabla_\perp^2+\frac{2k^2}{n_0}R_0P-\frac{k^2}{n_0}\gamma Pr^2\right)E(\mathbf{r},z)=0, \tag{2}$$

where $P=\int|U(\mathbf{r})|^2dxdy$ is the beam power, γ is a material constant, and R_0 is the maximum of $R(\mathbf{r})$. By letting $E(\mathbf{r},z)=U(\mathbf{r},z)\exp[i(k/n_0)R_0Pz]$ we finally obtain the model useful for wave propagation in HNN media [1]

$$(2ik\partial_z+\nabla_\perp^2-k^2a^2r^2)U(\mathbf{r},z)=0, \tag{3}$$

where a is a parameter that depends on the beam power P upon $a^2=\gamma P/n_0$.

Equation (3) is recognized to be the same as the equation that describes the propagation in a graded-index (GRIN) medium whose refractive index varies radially as $n(r)=n_0(1-a^2r^2/2)$. Because the physics of this problem is well under-

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stood [22], it is easy to translate it into the context of soliton propagation. The simplest solution of Eq. (3) was studied in Ref. [1] and corresponds indeed to the azimuthally symmetric Gaussian mode

$$G(\mathbf{r},z) = \frac{aq_0}{aq_0\cos(az) + \sin(az)} \exp\left[\frac{ikr^2}{2q(z)}\right], \tag{4}$$

where $q_0 = q(0)$ is the initial complex beam parameter whose dependence on z is given by

$$q(z) = \left(\frac{1}{a}\right)\frac{aq_0\cos(az) + \sin(az)}{\cos(az) - aq_0\sin(az)}. \tag{5}$$

From Eqs. (4) and (5), it is clear that this fundamental beam preserves its Gaussian shape but its width breathes sinusoidally (with pitch period $L=2\pi/a$) in propagation. For a critical power $P=P_c$ the beam diffraction can be balanced by self-focusing and then the beam width remains constant under propagation. This is the case of soliton propagation. By letting w_0 be the beam width, we see from Eq. (5) that soliton condition is reached when $q_0=-ikw_0^2/2=-i/a$ (or equivalently $P_c=4/\gamma^2k^2w_0^4$) for which $q(z)=-i/a$ becomes constant.

Equation (4) describes the axial propagation of a Gaussian beam in HNN media. An arbitrary initial field (not necessarily azimuthally symmetric about the origin) can be formally expressed as a superposition of Gaussian beams. To preserve the highly nonlocal condition we assume that the spatial extent of each constituent beam, as well as the distance between them, is negligible compared to the characteristic nonlocal response length. The centroid (i.e., the first-order moment) of the initial field defines the center of the induced parabolic medium and propagates in a straight line by virtue of the conservation of transverse momentum.

In attempting to obtain a more general description of the field evolution in HNN media we first note that Gaussian localization is essential to propagation in a HNN medium, then to solve Eq. (3) we introduce the following ansatz: $U(\mathbf{r},z)=\Psi(\mathbf{r},z)G(\mathbf{r},z)$, where $\Psi(\mathbf{r},z)$ is a modulating function to be determined. Substituting this ansatz into Eq. (3), we get the following equation for $\Psi(\mathbf{r},z)$:

$$2ik\partial_z\Psi + \nabla_\perp^2\Psi + \frac{2ik}{q(z)}(\nabla_\perp\Psi\cdot\mathbf{r}) = 0. \tag{6}$$

The standard approach to solve an equation like Eq. (6) consists of assuming a separable solution of the form $\Psi = f(x_1)g(x_2)h(z)$, where (x_1,x_2) are two particular orthogonal coordinates in the transverse plane. This procedure leads to pure soliton solutions described by Hermite-Gaussian and Laguerre-Gaussian functions [23]. To follow a coordinate-free approach rather than proposing solutions in a particular coordinate system, we will consider a general field of the form $\Psi(\mathbf{r},z)=W(\mathbf{r},z)\zeta(z)$, where $W(\mathbf{r},z)$ accounts for the transverse beam structure and admits the following general plane wave expansion:

$$W(\mathbf{r},z) = \int_{-\pi}^{\pi} A(\varphi)\exp[i\kappa(z)(x\cos\varphi + y\sin\varphi)]d\varphi, \tag{7}$$

where $\kappa(z)$ is the characteristic transverse wave number to be determined, and $A(\varphi)$ is the arbitrarily complex angular spectrum of the beam [24–26]. In assuming $W(\mathbf{r},z)$ as Eq. (7) we are setting the center of the parabolic medium, defined as the first moment of the intensity profile $|U|^2$, to be located at $\mathbf{r}=0$. Substitution of Eqs. (5) and the ansatz into Eq. (6) leads to $\zeta(z)=\exp[-i\sin(az)\kappa_0\kappa(z)/2ka]$, and

$$\kappa(z) = \frac{aq_0\kappa_0}{aq_0\cos(az) + \sin(az)}, \tag{8}$$

where $\kappa_0=\kappa(0)$ is the initial transverse wave number and, as expected, the function $W(\mathbf{r},z)$ must to satisfy the two-dimensional (2D) Helmholtz equation $\nabla_\perp^2 W+\kappa^2(z)W=0$.

Collecting the partial solutions provides the desired expression for $U(\mathbf{r},z)$, namely,

$$U(\mathbf{r},z) = \exp\left[-\frac{i\kappa_0\kappa(z)\sin(az)}{2ka}\right]G(\mathbf{r},z)W(\mathbf{r},\kappa). \tag{9}$$

Equation (9) permits an arbitrary HzG mode to be propagated in closed form through a HNN medium. The field $U(\mathbf{r},z)$ results from the product of three factors. The exponential factor is just a complex amplitude depending on z only. The Gaussian factor $G(\mathbf{r},z)$ ensures the transverse confinement and finite beam power. Whereas $G(\mathbf{r},z)$ includes only the possibility of circularly symmetric Gaussian spots and spherical phase fronts [1], the factor $W(\mathbf{r},\kappa)$ allows for the possibility of much more complicated transverse patterns. The inclusion of this term creates maxima and minima, and possibly beam nulls, in the amplitude distribution. Mathematically, $W(\mathbf{r},\kappa)$ is an arbitrary solution of the two-dimensional Helmholtz equation and physically can be associated to the transverse shape of an ideal nondiffracting beam, including, for example, profiles belonging Bessel, Mathieu, or parabolic nondiffracting beams [24–26]. In general, the shape of the HzG modes in HNN media will change under propagation because κ_0 and κ are not proportional to each other through a real factor, leading to different profiles of the function W. Nevertheless, κ and q vary periodically with a period $L=2\pi/a$, therefore the initial field self-reproduces after this distance.

III. PHYSICAL DISCUSSION

Physical insight into the propagation of HzG modes in HNN media is gained by observing that $U(\mathbf{r},z)$ can be viewed as a superposition of tilted Gaussian beams whose mean propagation axes follow the geometrical sinusoidal path around the optical axis predicted by the ray propagation theory in GRIN media [22]. In general the width of the constituent Gaussian beams will oscillate as field propagates, however, if the soliton condition is satisfied (i.e., $q_0 = -ikw_0^2/2=-i/a$) then each constituent Gaussian beam becomes a constant-width soliton. For this condition, Eqs. (5) and (7) reduce to $q(z)=-i/a$ and $\kappa(z)=\kappa_0\exp(-iaz)$, respec-

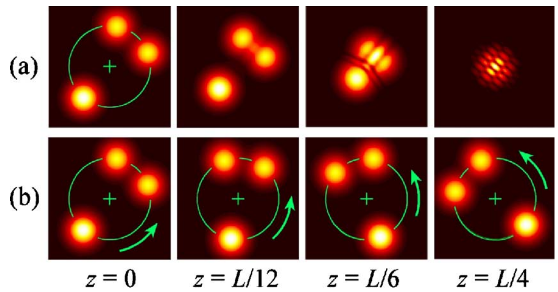

FIG. 1. (Color online) (a) Propagation and interaction of three solitons launched in parallel into a HNN medium with $a=1$. (b) Helical trajectories of the three solitons launched with the appropriate skew.

tively, and the expression for $U(\mathbf{r},z)$ takes the particularly simple form

$$U(\mathbf{r},z) = f(z)\exp(-r^2/w_0^2)W(\mathbf{r},\kappa), \tag{10}$$

where $f(z)=\exp[-iaz-i\kappa_0^2\sin(az)\exp(-iaz)/2ka]$ depends on z only, and the constant Gaussian width is related to beam power by $w_0^2=2/\gamma k\sqrt{P}$.

Equation (10) represents a general expression to describe the propagation of $(2+1)D$ solitons and their interactions in HNN media. The field $U(\mathbf{r},z)$ is characterized by two transverse characteristic lengths, namely, w_0 for the Gaussian envelope, and $1/\kappa_0$ for function W. The physical meaning of these parameters is important; whereas w_0 adjusts the width of the Gaussian modulation, the parameter κ_0 governs the oscillatory behavior of the function W in the transverse direction.

We identify two important cases. The first one occurs when $w_0 \gg 1/\kappa_0$, i.e., when the Gaussian width is much larger than the transverse beam oscillations. As an example, let us discuss the interaction of three solitons of total power P launched in parallel into a HNN medium as shown in Fig. 1(a). The solitons see a parabolic medium (with parameter $a=\sqrt{\gamma P/n_0}$) whose axis coincides with the center of the circle crossing over the soliton maxima, thus they are equally displaced from the optical axis. The field is given by Eq. (10) with $W=\Sigma_{j=1}^{3}A_j\exp[i\kappa r\cos(\theta-\theta_j)]$, where $A_j=\{1,1,3^{1/4}\}$, $\theta_j=\{20°,80°,230°\}$, $\kappa_0=-i2\pi$, and $w_0=1$. In propagation, the solitons will interact by undergoing sinusoidal trajectories about the optical axis and will suffer periodic collisions at positions $z_m=L(m+1/2)/2$, where $m=0,1,2,\ldots$, while maintaining its width constant. Note that the entire propagation of the three solitons is described in closed form by Eq. (10). Since the sinusoidal trajectories are independent of the individual initial phases, these affect the interference pattern at planes z_m only. By inducing an appropriate initial skew to the solitons, they will spiral about the axis of the parabolic medium, and will undergo rotation as propagate, as shown in Fig. 1(b). Interaction of a large number of solitons with different launching amplitudes and phases can be described in a similar way. More complex structures can be also modeled using a superposition of beams of the form (10) with different κ_0.

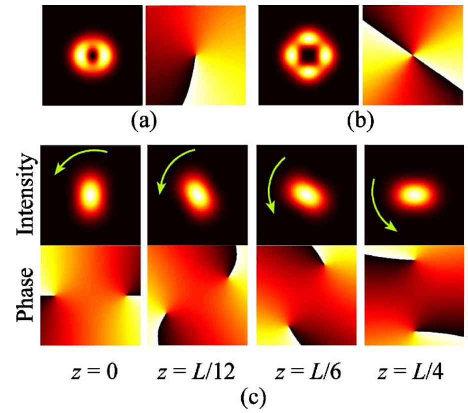

FIG. 2. (Color online) Quasi-invariant (a) dipole and (b) quadrupole vortex self-trapped HzG beam. (c) Rotating propagation of an elliptic vortex self-trapped HzG beam constructed with a helical Mathieu beam.

The second case occurs when $w_0 \ll 1/\kappa_0$, i.e., when the transverse beam oscillations are much larger than the Gaussian width. In this case, the variation in the field shape on propagation is very small ($<5\%$), making possible the formation of quasipropagation-invariant solutions. For example, Figs. 2(a) and 2(b) shows the intensities and phases of two vortex self-trapped beams carrying angular momentum. The fields are described by Eq. (10) with (a) $W(\mathbf{r},0)=J_1(\kappa_0 r)\times(\cos\theta+i0.75\sin\theta)$ where J_n is the nth order Bessel function, and (b) $W(\mathbf{r},0)=J_2(\kappa_0 r)[\cos(2\theta)+i0.75\sin(2\theta)]$. Both profiles are azimuthally asymmetric and propagate through the HNN medium maintaining its shape to within a very good approximation. The intensity patterns in Figs. 2(a) and 2(b) closely resemble those reported in Refs. [16] and [21] for dipole solitons and four-peaks azimuthons.

The result in Eq. (10) is general in the sense that function W is an arbitrary solution of the two-dimensional Helmholtz equation, and thus it is not associated to a particular coordinate system. This fact allows us to express a large variety of self-trapped modes and phenomena in closed and elegant form. For example, in Fig. 2(c) we show the rotating propagation of an elliptic vortex self-trapped beam with topological charge of 2 built with the helical Mathieu beam [27] in elliptical coordinates (ξ,η), $W(\mathbf{r},z)=\mathrm{Je}_m(\xi;\varepsilon)\mathrm{ce}_m(\eta;\varepsilon)+i\mathrm{Jo}_m(\xi;\varepsilon)\mathrm{se}_m(\eta;\varepsilon)$, where Je_m and Jo_m are the even and odd radial Mathieu functions with ellipticity ε, and ce_m and se_m are the even and odd angular Mathieu functions, respectively. Whereas the ellipticity of the beam can be adjusted with the parameter ε of the Mathieu functions, the rotation is determined uniquely by the strength of the power and the beam width through $a=\sqrt{\gamma P/n_0}$.

The solutions presented in this paper are strictly valid for the limiting case of propagation in HNN media. It is natural to ask whether it is possible to obtain these solutions in realistic nonlocal media if the HNN limit fails. To investigate this case, we have solved numerically the nonlinear nonlocal Schrödinger equation [Eq. (1)] by applying an accurate Fourier-based split-step beam propagation method [16,17,21] and taking as initial shapes several HzG profiles [Eq. (10) with $z=0$]. We have used a nonlocal nonlinear response in

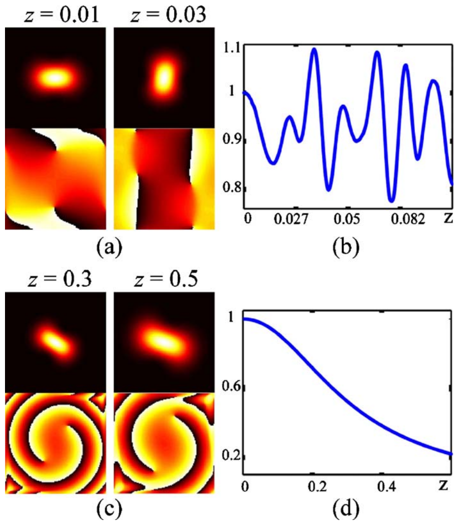

FIG. 3. (Color online) Propagation dynamics of an elliptic vortex self-trapped HzG beam in a nonlocal nonlinear medium. (a), (b) High nonlocality (the Gaussian width nonlocal response is three times larger than the maximum beam width). Here the beam remains self-trapped and many periods of rotation can be observed. The maximum value of the normalized intensity oscillates but remains within a finite range. (c) Intensity and phase profiles for low nonlocality (the Gaussian width nonlocal response is equal to the maximum beam width). (d) The beam diffracts decreasing its maximum normalized intensity and there is not significant intensity rotation.

the form of a normalized Gaussian function in Eq. (1) [11,19]. We first considered a very large nonlocality. The numerical analysis revealed that under appropriate conditions of enough degree of nonlocality, our numerical solutions behave very close to the theoretical prediction given by the highly nonlocal limit [Eq. (10)]. As the strength of the nonlocality is decreased, the beam begins to exhibit slight changes in its pitch period and its rotation velocity, as shown in Figs. 3(a) and 3(b), where we illustrate the propagation of the same rotating elliptic vortex HzG mode shown in Fig. 2. Although the intensity maximum oscillates stronger than the corresponding oscillations in a HNN medium [Fig. 3(b)], the beam essentially remains self-trapped. By decreasing the nonlocality even more, we found that there is a threshold where the initial profile will diffract without rotation, as shown in Figs. 3(c) and 3(d). Beyond this point, the HzG model cannot be considered as a useful model to describe nonlocal beam propagation. The stability analysis of HzG profiles propagating in nonlinear media with arbitrary nonlocality is still an open problem that is currently under study by the authors.

IV. CONCLUSIONS

In conclusion, we introduced an elegant description of $(2+1)D$ self-trapped beam phenomena in HNN media. The formalism is coordinate-free and offers a simple way of representing a variety of new kinds of self-trapped mode structures in a closed form, including, for example, noncircularly symmetric patterns, rotating patterns, and necklace beams. Our results shed light on the connection between soliton propagation in HNN media [1–4,6,7], the linear propagation in GRIN media [22], and the nondiffracting propagation in free space [24]. The model can be applied in nonlinear propagation in different physical systems featuring highly nonlocal response (e.g., thermal and liquid crystal media [3,4,6,7]) assuming the natural restrictions that effects of boundaries and anisotropy are not taken into account in the model, and that the center of mass of the beam must be located at the origin as the beam propagates.

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