Fractionalization of optical beams: I. Planar analysis

Julio C. Gutiérrez-Vega

Photonics and Mathematical Optics Group, Tecnológico de Monterrey, Monterrey, México 64849
juliocesar@itesm.mx

Received January 10, 2007; revised February 25, 2007; accepted February 26, 2007;
posted March 14, 2007 (Doc. ID 78876); published May 10, 2007

We apply the tools of the fractional calculus to introduce new fractional-order beam solutions to the paraxial wave equation that can be regarded as intermediate solutions between the known integral-order solutions. We restrict our attention to the fractionalization of the elegant and standard Hermite–Gaussian beams.

OCIS codes: 260.1960, 350.5500, 050.1960, 000.3860.

The role of the fundamental Gaussian, Hermite–Gaussian (HG), Laguerre–Gaussian, and Ince-Gaussian beams in the paraxial theory of optical beam propagation is well established [1,2]. For example, whereas the standard form of these beams is crucial in the theory of stable resonators [1], the elegant form, introduced by Siegman [3], is important in the paraxial theory of multipole complex-source point solutions of the Helmholtz equation [4,5]. An important property is that higher-order beam modes can be derived by acting with differential operators on the fundamental Gaussian beam [6]. The basis for such a construction is the following theorem: let u be a solution of the linear operator L (i.e., $Lu=0$); if another linear operator D commutes with L, then the function Du is also a solution of L. In particular, for a solution of the paraxial wave equation (PWE), its Cartesian derivatives are also solutions of the same equation.

On the other hand, in recent years fractional calculus has had applications in a number of different areas, such as fractal theory, polymers, and electrodynamics, to name a few [7–10]. The fractional calculus is the name for the theory of derivatives and integrals of arbitrary noninteger or even complex order [7,8]. Optics is a field in which the use of conventional calculus plays a major role, and it is of interest to see how fractional calculus may offer useful mathematical tools in this field. For example, fractionalization of the Fourier transform and its applications has been already studied by several researchers [11,12].

Following this line of interest, we have been exploring the possibility of extending the idea of fractional derivatives to fractionalization of Gaussian beams. In this Letter we introduce new fractional-order beam solutions to the PWE that can be regarded as intermediate solutions between the known integral-order solutions. We restrict our attention here to the planar geometry, which leads to the fractionalization of the HG beams. The propagation properties of the new solutions are discussed.

We begin the analysis by writing the complex field amplitude of the normalized one-dimensional fundamental Gaussian beam propagating in the positive z direction,

$$u_0(x,z)=\frac{(2/\pi)^{1/4}}{\sqrt{\mu w_0}}\exp\left(-\frac{x^2}{\mu w_0^2}\right), \tag{1}$$

where w_0 is the beam width at the waist plane $z=0$, and $\mu\equiv\mu(z)=1+iz/z_R$, with $z_R=kw_0^2/2$ being the Rayleigh distance and k the wavenumber. Since in rectangular coordinates the beam solutions can be separated into products of identical solutions in the x and y directions, for simplicity we consider solutions in only one rectangular coordinate and bring in the other coordinate by analogy. The field in Eq. (1) carries unit power (i.e., $\int_{-\infty}^{\infty}|u_0|^2\mathrm{d}x=1$) and satisfies the PWE $(\partial^2/\partial x^2+i2k\partial/\partial z)u=0$.

It is known that higher-order elegant and standard HG beams can be derived from the fundamental Gaussian beam by the repeated application of the differential operators $D=\partial/\partial x$ and $Q=(x/w_0)-(w_0\mu^*/2)\partial/\partial x$, respectively [6]. In particular, for the elegant HG beams of mth order we have

$$u_m=\frac{\partial^m u_0}{\partial x^m}\propto\frac{1}{\mu^{m/2}}H_m\left(\frac{x}{\sqrt{\mu}w_0}\right)u_0(x,z), \tag{2}$$

where H_m is the mth-order Hermite polynomial.

To explore the application of fractional calculus in beam theory, let us consider the definition of the Riemann–Liouville αth-order fractional derivative of the function $f(x)$ with respect to x based on the lower limit of integration a [7,8]:

$${}_a\mathrm{D}_x^\alpha f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}n}{\mathrm{d}x^n}\int_a^x(x-\xi)^{n-\alpha-1}f(\xi)\mathrm{d}\xi, \tag{3}$$

where n is the smallest integer exceeding α [i.e., $(n-1)\leqslant\alpha<n$], $\mathrm{d}/\mathrm{d}x$ denotes the ordinary derivative, and Γ is the Gamma function. Unlike the ordinary integer-order derivatives, the fractional derivative depends on the value of the lower limit of integration a. For positive and integer values of $\alpha=m$, Eq. (3) reduces to the ordinary integer-order derivative $\mathrm{d}^m/\mathrm{d}x^m$, independently of the value of a. For our purposes, the function $f(x)$ will be the Gaussian function whose transverse extent is infinite; thus we set

$a=-\infty$, and thus we shall adopt the simpler notation D^{α} to denote the αth-order fractional derivative operator with respect to x.

Like the integer-order differentiation, the fractional differentiation is a linear operation, i.e., $D^{\alpha}(c_1 f + c_2 g) = c_1 D^{\alpha} f + c_2 D^{\alpha} g$, where c_1 and c_2 are constants. In Eq. (3), if at the lower limit ($x=-\infty$) the first $(n-1)$ integral derivatives of $f(x)$ vanish (and this is the case for a Gaussian function), then the fractional derivative operator D^{α} commutes with the integral derivative $\mathrm{d}^m/\mathrm{d}x^m$, that is [8],

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m} D^{\alpha} = D^{\alpha} \frac{\mathrm{d}^m}{\mathrm{d}x^m} = D^{\alpha+m}. \quad (4)$$

It is now clear that the fractional operator D^{α} commutes with the operator of the PWE; therefore we expect that the action of D^{α} on any solution of the PWE will give a new solution of the same equation.

Fractionalization of the Gaussian beam is obtained by substituting Eq. (1) into Eq. (3). Although this procedure is formally correct, the evaluation of the integral and the further n-fold differentiation appear to be cumbersome. Instead of dealing directly with the evaluation of Eq. (3), we take advantage of the following theorem of the fractional calculus: If $\tilde{f}(k_x) = (2\pi)^{-1/2} \int f(x) \exp(-ik_x x) \mathrm{d}x$ is the Fourier transform of the function $f(x)$, then the Fourier transform of the αth-order derivative $D^{\alpha} f$ is $(ik_x)^{\alpha} \tilde{f}(k_x)$, for any fractional or even complex order α. By observing first that the Fourier transform of the Gaussian beam, Eq. (1), is $\tilde{u}_0 = (w_0^2/2\pi)^{1/4} \exp(-\mu k_x^2 w_0^2/4)$, we obtain

$$u_{\alpha}(x,z) = D^{\alpha} u_0(x,z) = \sqrt{w_0}(2\pi)^{-3/4} \times \int_{-\infty}^{\infty} (ik_x)^{\alpha} \exp\left(-\frac{\mu k_x^2 w_0^2}{4} + ik_x x\right) \mathrm{d}k_x. \quad (5)$$

The evaluation of this integral yields the expression for the fractional Gaussian beam [13]

$$u_{\alpha}(x,z) = \sqrt{\frac{N_{\alpha}}{\mu^{\alpha+1}}} \exp\left(-\frac{x^2}{\mu w_0^2}\right) P_{\alpha}\left(\frac{x}{\sqrt{\mu} w_0}\right), \quad (6)$$

with $P_{\alpha}(X)$ given by

$$P_{\alpha}(X) \equiv \cos\left(\frac{\pi}{2}\alpha\right) \Gamma\left(\frac{1+\alpha}{2}\right) \Phi\left(-\frac{\alpha}{2}, \frac{1}{2}; X^2\right) - \sin\left(\frac{\pi}{2}\alpha\right) \alpha \Gamma\left(\frac{\alpha}{2}\right) X \Phi\left(\frac{1-\alpha}{2}, \frac{3}{2}; X^2\right), \quad (7)$$

where $\Phi(a,b;z)$ is the Kummer confluent hypergeometric function (CHF) [13] [often also denoted ${}_1F_1(a,b;z)$], and $N_{\alpha} = 2^{\alpha+1/2}/\pi\Gamma(\alpha+1/2) w_0$ is the normalization constant that ensures unit power, i.e., $\int_{-\infty}^{\infty} |u_{\alpha}|^2 \mathrm{d}x = 1$.

Equation (6) is an exact solution of the PWE and is the main result of this study. It constitutes the fractionalization of the fundamental Gaussian beam and effectively corresponds to the close evaluation of the formal definition in Eq. (3) for a Gaussian function. At the waist plane $\mu=1$, $u_{\alpha}(x,z)$ becomes a purely real function. The behavior of $u_{\alpha}(x,0)$ as a function of x is depicted in Fig. 1 for the range $0 \leq \alpha \leq 12$. The Gaussian factor ensures the physical requirement that the field amplitude vanish for $|x|$ arbitrarily large and that the beam be square integrable. The function P_{α} creates maxima, minima, and beam nulls in the amplitude distribution as α increases. In particular, $u_{\alpha}(x,0)$ has m zeros when α falls in the interval $(m-1) < \alpha \leq m$. From the theory of the CHFs [13], it is possible to show that when α becomes a positive integer m, then $P_m(X) = \sqrt{\pi}(-1/2)^m H_m(X)$ and thus Eq. (6) reduces to Eq. (2) for the elegant HG beams (shown as solid curves in Fig. 1). So, effectively, the integer-order solutions have been smoothly connected by varying the order of fractional differentiation of the Gaussian beam.

While the fractional beam $u_{\alpha}(x,z)$ at the waist plane $z=0$ is purely real, outside this plane it becomes complex, leading to a continuous variation of the transverse pattern. This effect is illustrated in Fig. 2, where we show the propagation of a two-dimensional fractional beam constructed with the product $U_{\alpha_x,\alpha_y}(\mathbf{r}) = u_{\alpha_x}(x,z;w_{0x}) u_{\alpha_y}(y,z;w_{0y})$. Whereas the even and odd integer-order solutions are symmetrical and antisymmetrical about the origin, respectively, the fractional solutions do not have a definite parity.

Mathematical insight into the fractional beams is gained by using the identity $\Phi(a,b;z) = \exp(z)\Phi(b-a,b;-z)$ to rewrite Eq. (6) as

$$u_{\alpha}(\zeta) = A\Phi\left(\sigma, \frac{1}{2}; \zeta\right) + B\zeta^{1/2}\Phi\left(\sigma + \frac{1}{2}, \frac{3}{2}; \zeta\right), \quad (8)$$

where $\zeta \equiv -\gamma x^2$, $\sigma \equiv (1+\alpha)/2$, $\gamma(z) \equiv 1/w_0^2 \mu$, and the constants $A = (N_{\alpha}\mu^{-\alpha-1})^{1/2} \cos[(\pi/2)\alpha] \Gamma((1+\alpha)/2)$ and $B = -(N_{\alpha}\mu^{-\alpha-1}\gamma)^{1/2} \sin[(\pi/2)\alpha] \alpha \Gamma(\alpha/2)$. Equation (8) is identified as a linear superposition of the two independent even and odd solutions of the confluent hypergeometric equation $[\zeta \mathrm{d}^2/\mathrm{d}\zeta^2 + (\frac{1}{2} - \zeta)\mathrm{d}/\mathrm{d}\zeta - \sigma] u_{\alpha}(\zeta) = 0$. By re-expressing this differential equation in terms of the original variables, we determine that, as functions of x, the fractional function $u_{\alpha}(x,z)$ in Eq.

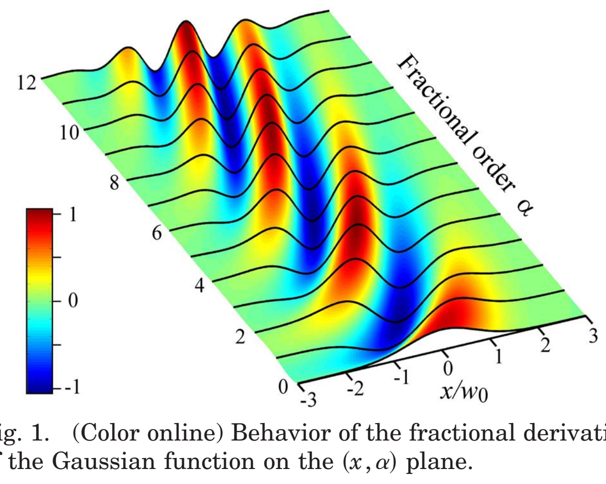

Fig. 1. (Color online) Behavior of the fractional derivative of the Gaussian function on the (x, α) plane.

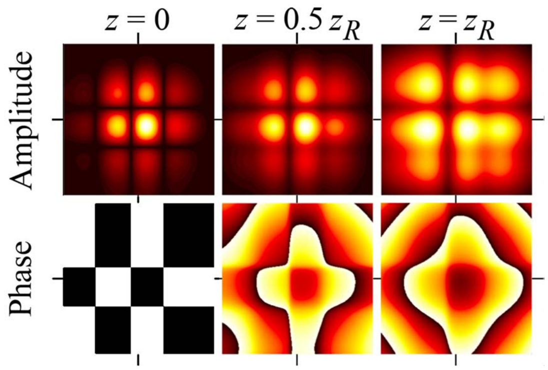

Fig. 2. (Color online) Amplitude and phase of the beam $U_{\alpha_x,\alpha_y}(\mathbf{r})$ at $z=\{0,0.5z_R,z_R\}$ for $\alpha_x=2.5$, $\alpha_y=1.6$, and $w_{0x}=w_{0y}$. Square image size of $6w_0\times 6w_0$.

(6) is solution of the ordinary differential equation

$$Tu_\alpha = \left[\frac{\mathrm{d}}{\mathrm{d}x^2} + 2\gamma x\frac{\mathrm{d}}{\mathrm{d}x} + 2\gamma(1+\alpha)\right]u_\alpha = 0, \qquad (9)$$

at any z plane. Equation (9) may be seen as a generalized form of the Hermite differential equation $[\mathrm{d}^2/\mathrm{d}x^2 - 2x\mathrm{d}/\mathrm{d}x + 2m]H_m(x)=0$.

The effect of applying the fractional derivative operator D^β to the function $u_\alpha(x)$ is determined by noting that the commutator of D^β and the operator T in Eq. (9) is given by $[D^\beta,T]=2\gamma\beta D^\beta$. It follows that the new function $D^\beta u_\alpha$ satisfies the same differential equation (9) with the parameter α changed to $\alpha+\beta$. We then conclude that the effect of the fractional differentiation on u_α is simply to change the order while retaining the original functional form, i.e., $D^\beta u_\alpha = u_{\alpha+\beta}$.

Since Eq. (9) is not a self-adjoint equation, its solutions u_α do not form an orthonormal set. The adjoint equation to Eq. (9) is found to be $(\mathrm{d}^2/\mathrm{d}x^2 - 2\gamma^* x\mathrm{d}/\mathrm{d}x + 2\alpha\gamma^*)\hat{u}_\alpha = 0$, whose solution is given by $\hat{u}_\alpha \propto A\Phi(-\alpha/2,\tfrac{1}{2};\gamma^* x^2) + Bx\Phi((1-\alpha)/2,\tfrac{3}{2};\gamma^* x^2)$. Note that there is no Gaussian factor associated with the adjoint functions. Applying the theory of the Kummer CHFs [13], it is possible to show that the biorthogonality integral $\int_{-\infty}^{\infty} u_\alpha \hat{u}_\beta^* \mathrm{d}x$ between u_α and its adjoint functions $\hat{u}_\beta$ diverges for arbitrary values of α and β unless both α and β become integer numbers, i.e., the biorthogonal relation for the elegant HG beams [3]. We then conclude that it is not possible to formulate an orthogonality relation for functions u_α with arbitrary α in the infinite domain $-\infty < x < \infty$. Nevertheless, one may be established in a finite domain by converting Eq. (9) into a self-adjoint form and applying the Sturm–Liouville theory.

Until now, we have explored the fractionalization of the elegant HG beams. We now discuss the fractionalization of the standard HG beams [1]. It is known [6] that on mth-order standard HG beam $u_m^S = (\mu^*/\mu)^{m/2} u_0 H_m(\sqrt{2}x/w_0|\mu|)$ is obtained by acting m times the operator $Q = x/w_0 - (w_0\mu^*/2)\mathrm{d}/\mathrm{d}x$ on the fundamental Gaussian beam u_0 given by Eq. (1). We look for fractional beams resulting from the operation $Q^\alpha u_0$ where α is an arbitrary positive number. Taking advantage of the operator identity $(\partial/\partial x - 2cx)^\alpha = \exp(cx^2)D^\alpha \exp(-cx^2)$ and the fractional derivative of the Gaussian function in Eq. (6), we determine the expression for the standard HG beams of αth order,

$$u_\alpha^S(x,z) = (\mu^*/\mu)^{\alpha/2} u_0 P_\alpha(\sqrt{2}x/w_0|\mu|). \qquad (10)$$

Like the standard HG beams of integral order [1] the fractional modes are shape invariant except for a scaling factor. Although Eq. (10) is an exact solution of the PWE, a numerical evaluation shows that, except for integer values of α, it diverges as $|x|\to\infty$. Recalling the analogy between wave propagation in graded-index media and the quantum harmonic oscillator [11], we remark that this result is consistent with the known fact that the eigenfunctions of the harmonic oscillator must be quantized. To get physically acceptable solutions resulting from the action of the fractional operator Q^α on u_0, we then need to consider problems involving finite domains (e.g., planar waveguides filled with decentered graded-index media), rather than infinite domains.

In conclusion, fractional-order solutions of the PWE were derived by operation of the fractional derivative on the fundamental Gaussian beam. The new solutions constitute continuous transition modes between the integer-order elegant HG beams. Fractionalization of standard HG beams acquires physical meaning in problems involving a finite domain. The Fourier transform of the fractional beam was found in closed form as well. An interesting issue to pursue is whether the concept of fractional beams can be extended to intermediate modes in planar wave-guide structures and multipole complex-source point solutions of the Helmholtz equation [10].

We thank Consejo Nacional de Ciencia y Tecnología (grant 42808) and Tecnológico de Monterrey (grant CAT007).

References

1. A. E. Siegman, *Lasers* (University Science, 1986).
2. M. A. Bandres and J. C. Gutiérrez-Vega, Opt. Lett. **29**, 144 (2004).
3. A. E. Siegman, J. Opt. Soc. Am. **63**, 1093 (1973).
4. S. Y. Shin and L. B. Felsen, J. Opt. Soc. Am. **67**, 699 (1977).
5. S. R. Seshadri, Opt. Lett. **28**, 595 (2003).
6. J. Enderlein and F. Pampaloni, J. Opt. Soc. Am. A **21**, 1553 (2004).
7. I. Podlubny, *Fractional Differential Equations* (Academic, 1999).
8. K. Oldham and J. Spanier, *The Fractional Calculus* (Academic, 1974).
9. N. Engheta, IEEE Antennas Propag. Mag. **39**, 35 (1997).
10. N. Engheta, IEEE Trans. Antennas Propag. **44**, 554 (1996).
11. H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform* (Wiley, 2001).
12. A. W. Lohmann, D. Mendlovic, and Z. Zalevsky, Prog. Opt. **38**, 265 (1998).
13. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, 1964), Chap. 13.