Comment on "Eigenfields and output beams of an unstable Bessel–Gauss resonator"

Raul I. Hernandez-Aranda and Julio C. Gutiérrez-Vega

We comment on a recent paper by D. Ling *et al.* [Appl. Opt. **45**, 4102 (2006)]. In that paper, the authors adopted the entire matrix formalism that we established in a previous work [J. Opt. Soc. Am. A **22**, 1909 (2005)] for finding the eigenmodes of an unstable Bessel resonator. Nevertheless, the results are inaccurate mainly because (a) it was overlooked that light crosses through the axicon twice in a complete round trip and (b) the numerical method used to evaluate the diffraction integral equations cannot resolve the eigenvalues and eigenfields for the given resonator configuration.

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In a recent paper[1] Ling *et al.* claimed the introduction of a transfer-matrix method for finding the eigenmodes of an unstable Bessel resonator (UBR). We have been involved for some time in studies concerning properties of modes of axicon-based Bessel resonators, including both theoretical and experimental aspects.[2–4] Since Ling *et al.* analyzed a resonator configuration that is identical (i.e., in every numerical value for all the optical elements) to the configuration that we analyzed previously in Ref. 2, we would expect both papers to yield very similar results. However, after a first view of Ref. 1, we realized that the reported modal patterns, the mode losses, the mode ordering by loss, and the intracavity field distributions exhibit evident inconsistencies. The findings in Ref. 1 encouraged us to try some numerical procedures to reproduce the results reported by Ling *et al.* In this Comment, we argue that the results reported in Ref. 1 are inaccurate.

In Ref. 1 the authors fail to consider that in a complete round trip through the UBR the light crosses through the refractive axicon twice.[2–8] This is a contradictory situation, because this fact was pointed out throughout the paper, but it was overlooked in the derivation of the matrix equations. To explain the origin of the error, we first recall that a refractive axicon of refraction n and wedge angle α are characterized by the linear radial phase factor

$$T(r) = \exp(-ik\theta_0 r), \tag{1}$$

where $\theta_0 = (n - 1)\alpha$ and k is the wavenumber. In a single round trip a light ray crosses the axicon twice; therefore in the corresponding lens-guide equivalent system the element turns into a double axicon with transmittance $T^2(r) = \exp(-i2k\theta_0 r)$. To perform a ray-tracing analysis of the axicon-based resonators, in Refs. 2 and 3 we introduced an inhomogeneous *ABCD* matrix for the double axicon given by

$$\mathbf{M}_{\mathrm{dax}} = \begin{bmatrix} 1 & 0 \\ -2\theta_0/r & 1 \end{bmatrix}, \tag{2}$$

which was particularly useful for determining the transfer matrix for the complete cavity [see Eq. (4) in Ref. 2], and then for finding the stable self-reproducing ray trajectories. It is important to note that, because the radial phase variation of the axicon is not quadratic but linear, the matrix $\mathbf{M}_{\mathrm{dax}}$ cannot be applied directly to the Huygens diffraction integral for propagation through a first-order *ABCD* optical system.

In Ref. 1 the authors adopted exactly the same *ABCD* ray matrix formulation established in Ref. 2 and used the two partial one-way transfer matrices, their Eqs. (6) and (7), to perform a wave-optics analysis of the UBR. The crucial error was that, to solve their eigenvalue equation (13), the authors directly inserted the element $A = 1 - 2\theta_0 L/r_1$ of matrix $\mathbf{T}_1$ of Eq. (6) into the transfer matrix of Eq. (11), and con-

R. Hernandez-Aranda (raul.aranda@itesm.mx) and J. Gutiérrez-Vega (juliocesar@itesm.mx) are with the Photonics and Mathematical Optics Group, Garza Sada 2501, Monterrey, Mexico 64849.

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sequently, into the Huygens diffraction integral, Eq. (4). At this point, it is important to remark that there are several typos in Eqs. (4)–(7) of Ref. 1; specifically, the upper limits of the integrals in Eqs. (4) and (5) should be a_1 and a_2, respectively, whereas the subscripts of the *ABCD* elements in the matrices of Eqs. (6) and (7) should be 1 and 2, respectively.

The consequences of the error are readily appreciated when writing the Huygens integral for propagation of the field $E_1(r_1)$ through an *ABCD* system [Eq. (4) of Ref. 1]:

$$E_2(r_2) = \frac{(-i)^{n+1} k e^{ikL}}{B_1} \int_0^{a_1} E_1(r_1) J_n\left(\frac{k r_1 r_2}{B_1}\right) \times \exp\left[\frac{ik}{2B_1}(A_1 {r_1}^2 + D_1 {r_2}^2)\right] r_1 \mathrm{d}r_1. \tag{3}$$

Inserting now the elements of matrix $\mathbf{T}_1$ of Eq. (6) into Eq. (3), for the exponential term within the integral we obtain

$$\exp\left\{\frac{ik}{2L}\left[\left(1 - \frac{2\theta_0 L}{r_1}\right){r_1}^2 + {r_2}^2\right]\right\} = \exp(-ik\theta_0 r_1) \times \exp\left[\frac{ik}{2L}({r_1}^2 + {r_2}^2)\right]. \tag{4}$$

Consequently the propagated field E_2 becomes

$$E_2(r_2) = \frac{(-i)^{n+1} k e^{ikL}}{L} \int_0^{a_1} [E_1(r_1)\exp(-ik\theta_0 r_1)] J_n\left(\frac{k r_1 r_2}{L}\right) \times \exp\left[\frac{ik}{2L}({r_1}^2 + {r_2}^2)\right] r_1 \mathrm{d}r_1. \tag{5}$$

This integral equation represents simply the free-space propagation of the initial field $E_1(r_1)\exp(-ik\theta_0 r_1)$ through a distance L. By noting that the term $\exp(-ik\theta_0 r_1)$ is indeed the transmittance of a single axicon, it is evident now that the transfer matrix derived in Ref. 1 considers that the light crosses once, instead of twice, through the axicon in a complete round trip. The correct procedure to handle the axicon for the resonator under consideration consists in extracting its transmittance from the *ABCD* matrix of the whole system, as explained in Ref. 2. The error leads to the fact that the cavity now sees an equivalent axicon with wedge angle divided by two, and eventually to the violation of the condition[3]

$$L = \frac{a}{2(n-1)\alpha}, \tag{6}$$

which relates cavity length L to wedge angle α, the refractive index n, and the aperture radius a of the axicon.

Since it is natural to suspect that the error in Eq. (11) was merely a misprint, we computed the mode patterns and mode losses of the UBR by deliberately considering that light crosses the axicon only once. We employed two independent approaches: the transfer-matrix method explained in Ref. 2, and a Fox–Li method based on a Hankel transform propagator.[9,10] Both methods yielded the same erroneous results reported in Table 1 of Ref. 1. On the other hand, we also implemented the algorithm proposed in Ref. 1, but now considering correctly that light crosses twice through the axicon; in this way we obtained the correct losses and field patterns reported in Ref. 2.

The implementation of Eqs. (11) and (13) in Ref. 1 leads to several inconsistencies. For example:

- In Table 1, it is reported that the lowest-order modes of an UBR are the (0, 0) and (2, 0) modes, implying that the mode (2, 0) is more dominant than the mode (1, 0). These results are incompatible with those of existing literature and experimental evidence[2–7,11] that show that the expected mode ordering is (0, 0), (1, 0), (2, 0), and so on.
- The radial amplitudes across the output mirror are depicted in Fig. 5 of Ref. 1 for the modes (0, 0) and (2, 0). Although the plots show some radial oscillations, as expected for a fully axisymmetric resonator, these curves are inaccurate because they do not fit the expected Bessel oscillations at the output plane, in particular its radial frequency. For axicon-based resonators[2,3] the transverse wavenumber k_t of the expected Bessel modulation $J_n(k_t r)$ is determined exclusively by the parameters of the axicon through $k_t = k \sin[(n-1)\alpha]$. For the particular data used in Ref. 1 (which are exactly the same that we used in Ref. 2) we obtain $k_t = 7241\ \mathrm{m}^{-1}$, implying that the width of the Bessel rings is approximately 0.43 mm. For the output range $[0 < r < 5\ \mathrm{mm}]$ shown in Fig. 5 we would expect to see about 11 or 12 Bessel rings, as shown in Figs. 2 to 4 of Ref. 2.

To evaluate numerically the diffraction integral equations in Ref. 1, the authors applied a simple Riemann sum based on dividing the integration range into $M = 256$ equally spaced points. However, it is known that the applicability of the Riemann sum is limited to smooth and well-behaved integrands[12]; consequently, in situations where the field variations become faster, the accuracy of the algorithm proposed in Ref. 1 decreases dramatically. The advantages of Gauss–Legendre quadrature over more rudimentary numerical methods of integration for extracting eigenmodes of optical resonators using matrix methods have long been well established.[13–15] The intracavity field distributions and the output beams shown in Figs. 6–9 of Ref. 1 are also inaccurate because the initial fields (at $z = 0$ in all plots) cannot propagate in such a way that their overall intensities increase hundreds of times without violating the energy conservation principle. A correct intracavity field distribution of an UBR is shown in Fig. 5 of Ref. 2, where we can identify the conical region where the output beam is formed and preserves invariance and the radial Bessel oscillations whose amplitude and frequency agree with the theoretical model.

As a concluding comment, we do not see a reason to claim the introduction of a new transfer-matrix method for extracting the eigenmodes of the UBR, because in Ref. 2 we had already applied the same matrix method for analyzing exactly the same problem. The discrepancies between Refs. 1 and 2 are due to conceptual and numerical errors in the analysis of the UBR in Ref. 1. The use of a Riemann sum, instead of improving, decreases the accuracy of the original Gauss–Legendre method applied in Ref. 2.

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