Generalized Helmholtz–Gauss beam and its transformation by paraxial optical systems

Manuel Guizar-Sicairos

The Institute of Optics, University of Rochester, Rochester, New York 14627

Julio C. Gutiérrez-Vega

Photonics and Mathematical Optics Group, Tecnológico de Monterrey, Monterrey, México 64849

Received June 15, 2006; revised July 25, 2006; accepted July 26, 2006;
posted July 26, 2006 (Doc. ID 72034); published September 11, 2006

We introduce the generalized Helmholtz–Gauss (gHzG) beam and analyze its propagation through optical systems described by *ABCD* matrices with real and complex elements. The transverse mathematical structure of the gHzG beam is form invariant under paraxial transformations and reduces to those of ordinary HzG and modified HzG beams as special cases. We derive a closed-form expression for the fractional Fourier transform of gHzG beams. © 2006 Optical Society of America

OCIS codes: 260.1960, 350.5500, 140.3300, 050.1960.

The term Helmholtz–Gauss (HzG) beam was introduced recently to refer to a paraxial field whose disturbance at the plane $z=0$ reduces to the product of the transverse field of an arbitrary ideal nondiffracting beam (i.e., a solution of the 2D Helmholtz equation) and a radial Gaussian apodization.[1–3] HzG beams are square integrable and carry finite power; thus they constitute one of the simplest physical realizations of ideal nondiffracting beams.[4–6]

In this Letter we introduce a useful generalized form of the HzG beam. This form will be referred to as a generalized HzG (gHzG) beam and can handle propagation not only in free space but also in more general types of paraxial optical system characterized by complex *ABCD* matrices. It is found that the gHzG beams are a class of fields that exhibit the property of form invariance under paraxial optical transformations. We derive a closed-form expression for the fractional Fourier transform of the gHzG and discuss its meaningful special cases. This work consolidates and extends previous studies on Bessel–Gauss beams propagating through *ABCD* systems.[7–11]

We commence the analysis by writing the complex field amplitude of a gHzG beam at the input plane $z=z_1$ of a radially symmetrical first-order optical *ABCD* system

$$U_1(\mathbf{r}_1)=\exp\left(\frac{ikr_1^2}{2q_1}\right)W(\mathbf{r}_1;\kappa_1), \qquad q_1''<0, \tag{1}$$

where $\mathbf{r}_1=(x_1,y_1)=(r_1,\theta_1)$ denotes the transverse coordinates, k is the wavenumber, and single ($'$) and double ($''$) primes denote the real and the imaginary parts of a complex quantity, respectively.

Equation (1) results from the product of two functions, each depending on one parameter. The Gaussian apodization is characterized by a complex beam parameter $q_1=q_1'+iq_1''$. In assuming a complex q_1 we are allowing for the possibility that the Gaussian modulation has a converging ($q_1'<0$) or diverging ($q_1'>0$) spherical wavefront. For simplicity in dealing with the *ABCD* system, the parameter q_1, instead of the width $w_1=\sqrt{i2q_1/k}$, is used throughout this Letter.

The function $W(\mathbf{r}_1;\kappa_1)$ in Eq. (1) is a solution of the 2D Helmholtz equation $[\partial_{xx}+\partial_{yy}+\kappa_1^2]W=0$ and describes the transverse field of an ideal nondiffracting beam. It can be expanded as[4]

$$W(\mathbf{r}_1;\kappa_1)=\int_{-\pi}^{\pi}g(\phi)\exp[i\kappa_1 r_1\cos(\phi-\theta_1)]\mathrm{d}\phi, \tag{2}$$

where κ_1 and $g(\phi)$ are the transverse wavenumber and the angular spectrum of the ideal nondiffracting beam, respectively. Since $g(\phi)$ is arbitrary, an infinite number of profiles can be obtained[1]; for example, $g(\phi)=\exp(im\phi)$ leads to the mth-order Bessel beams $W=J_m(\kappa r)\exp(im\theta)$, whereas $g(\phi)=\mathrm{ce}_m(\phi,\varepsilon)$ leads to the mth-order even Mathieu beams $W=\mathrm{Je}_m(\xi,\epsilon)\mathrm{ce}_m(\eta,\epsilon)$, where $\mathrm{Je}_m(\cdot)$ and $\mathrm{ce}_m(\cdot)$ are the radial and angular even Mathieu functions, respectively.[6] A gHzG beam tends to a pure nondiffracting beam when $|q_1''|\to\infty$ and to a Gaussian beam when $\kappa_1\to 0$.

In the traditional approach to nondiffracting beams, the wavenumber κ_1 in Eq. (2) is customarily assumed to be real and positive.[1,4] For generality, we will let $\kappa_1=\kappa_1'+i\kappa_1''$ be arbitrarily complex, allowing the possibility of having three main cases: (a) Real $\kappa_1=\kappa_1'$ leads to the ordinary HzG (oHzG) beams for which $W(\mathbf{r}_1;\kappa_1)$ is a purely oscillatory function. (b) Purely imaginary $\kappa_1=i\kappa_1''$ leads to evanescent functions $W(\mathbf{r}_1,\kappa_1'')$ that satisfy the modified Helmholtz equation $[\partial_{xx}+\partial_{yy}-\kappa_1''^2]W=0$. Because of this, we shall refer to this case as a modified HzG (mHzG) beams. Particular examples are given by the cosh-Gaussian and modified Bessel–Gauss beam.[10,11] (c) Complex κ_1 leads to our gHzG beams which, as we shall see, can be interpreted as beam solutions intermediate between oHzG and mHzG beams.

To find the transverse field of the gHzG beams at the output plane $z=z_2$ of the *ABCD* system we proceed by writing the Huygens diffraction integral[12]

$$U_2(\mathbf{r}_2) = \frac{k \exp(ik\zeta)}{i2\pi B} \int\int_{-\infty}^{\infty} U_1(\mathbf{r}_1) \times \exp\left[\frac{ik}{2B}(Ar_1^2 - 2\mathbf{r}_1 \cdot \mathbf{r}_2 + Dr_2^2)\right] \mathrm{d}^2\mathbf{r}_1, \quad (3)$$

where $\mathbf{r}_2=(x_2,y_2)=(r_2,\theta_2)$ denotes the transverse coordinates at the output plane and ζ is the optical path length from the input to the output plane measured along the optical axis. After substituting Eq. (1) into Eq. (3) and using Eq. (2), the integration can be performed by applying the changes of variables $x_j=u_j \cos \phi - v_j \sin \phi$ and $y_j=u_j \sin \phi + v_j \cos \phi$, for $j=1,2$. On returning to the original variables we obtain

$$U_2(\mathbf{r}_2) = \exp\left(\frac{\kappa_1 \kappa_2 B}{i2k}\right) \mathrm{GB}(\mathbf{r}_2,q_2) W(\mathbf{r}_2;\kappa_2), \quad (4)$$

where

$$\mathrm{GB}(\mathbf{r}_2,q_2) = \frac{\exp(ik\zeta)}{A + B/q_1} \exp\left(\frac{ikr_2^2}{2q_2}\right) \quad (5)$$

is the output field of a Gaussian beam with input parameter q_1 traveling axially through the *ABCD* system, and the transformation laws for q_1 and κ_1 from plane z_1 to plane z_2 are

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}, \qquad \kappa_2 = \frac{\kappa_1}{A + B/q_1}. \quad (6)$$

Equation (4) is the main result of this study. It permits an arbitrary gHzG beam to be propagated in closed form through a real or complex *ABCD* optical system. Apart from a complex amplitude factor, the output field has the same mathematical structure as the input field; thus the gHzG beams constitute a class of fields whose form is invariant under paraxial optical transformations. This form-invariance property does not have to be confused with the shape-invariance property of the Hermite–Gauss or Laguerre–Gauss beams, which preserve, except by a scaling factor, the same transverse shape under paraxial transformations. The shape of the gHzG beams will change because κ_1 and κ_2 are not proportional to each other through a real factor, leading to different profiles of the function W, and also because the parameters q_1 and κ_1 are transformed according to different laws.

Physical insight into the gHzG beams is gained by studying some particular cases. We consider first the free-space propagation along a distance $L=z_2-z_1$, which is represented by the matrix $[A,B;C,D]=[1,L;0,1]$. The input and output fields are given by Eqs. (1) and (4), where from Eqs. (6) the transformation laws become $q_2=q_1+L$ and $\kappa_2=\kappa_1 q_1/(q_1+L)$. Note that in this case the product $q_2\kappa_2=q_1\kappa_1$ remains constant under propagation. A useful physical picture of the gHzG beams in free space can be built up from Eqs. (1) and (2) by noting that each constituent plane wave of the function W is multiplied by the Gaussian apodization. In this way, gHzG beams can be viewed as a superposition of tilted Gaussian beams whose mean propagation axes lie on the surface of a double cone whose generatrix is found to be

$$r_{\mathrm{gen}} = \frac{|q_1|^2 \kappa_1''}{kq_1''} + \left(\frac{\kappa_1' q_1'' + \kappa_1'' q_1'}{kq_1''}\right)(z - z_1), \quad (7)$$

and whose amplitudes are modulated angularly by $g(\phi)$. The position of the waist planes of the constituent Gaussian beams coincides with the plane where the radial factor $\exp(ikr_2^2/2q_2)$ becomes a real Gaussian envelope, that is, $z_{\mathrm{waist}}=z_1-q_1'$. At z_{waist}, the parameter q becomes purely imaginary, $q_{\mathrm{waist}}=iq_1''$, whereas the wavenumber reduces to $\kappa_{\mathrm{waist}}=\kappa_1(1-iq_1'/q_1'')$.

The extent of the region where the constituent Gaussian beams interfere is maximized where their main propagation axes intersect, i.e., the vertex of the double cone. The evaluation of the condition $r_{\mathrm{gen}}=0$ yields $z_{\mathrm{vertex}}=z_1-|q_1|^2\kappa_1''/(\kappa_1'q_1''+\kappa_1''q_1')$. Now, at the vertex plane the transverse wavenumber κ becomes purely real, $\kappa_{\mathrm{vertex}}=\kappa_1'+\kappa_1''q_1'/q_1''$, with the consequence that at this plane the transverse amplitude profile belongs to the oHzG kind with $q_{\mathrm{vertex}}=\kappa_1 q_1/(\kappa_{\mathrm{vertex}})$.

In particular, oHzG beams[1] are a special case of Eq. (1), when $\kappa_1''=0$, $q_1'=0$, and $z_1=0$, leading to a cone generatrix given by $r_{\mathrm{gen}}=(\kappa_1'/k)z$. On the other side, the mHzG beams occur when $\kappa_1'=0$; if we additionally set $q_1'=0$, then $r_{\mathrm{gen}}=r_0=q_1''\kappa_1''/k$, and thus the mHzG beams may be viewed as a superposition of Gaussian beams whose axes are parallel to the z axis and lie on the surface of a circular cylinder of radius r_0; see Ref. 11 for the case of modified Bessel–Gauss beams.

Further understanding of the gHzG beam can be gained by determining its fractional Fourier transform. To do this, we recall that the propagation over a distance $L=z_2-z_1=p\pi a/2$ in a graded refractive-index (GRIN) medium with index variation $n(r)=n_0(1-r^2/2a^2)$ results in the pth Fractional Fourier transform.[13] The *ABCD* matrix from plane z_1 to plane z_2 is given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos(p\pi/2) & a\sin(p\pi/2) \\ -\sin(p\pi/2)/a & \cos(p\pi/2) \end{bmatrix}. \quad (8)$$

The output field is described by Eq. (4), provided that

$$q_2 = a\frac{q_1\cos(p\pi/2) + a\sin(p\pi/2)}{-q_1\sin(p\pi/2) + a\cos(p\pi/2)}, \quad (9a)$$

$$\kappa_2 = \frac{\kappa_1 q_1}{q_1\cos(p\pi/2) + a\sin(p\pi/2)}. \quad (9b)$$

By setting $p=1$ in Eq. (8) we get the *ABCD* matrix $[0,a;-1/a,0]$ corresponding to the conventional Fourier transform. This is indeed the matrix transformation from the first to the second focal plane of a converging thin lens of focal length a. Let us assume that the input field of this Fourier transformer is an oHzG profile (i.e., $\kappa_1''=0$) with real Gaussian apodization (i.e., $q_1'=0$). From Eqs. (9) we see that both parameters $q_2=ia^2/q_1''$ and $\kappa_2=i\kappa_1'q_1''/a$ become purely imaginary. It is now evident that, if an oHzG profile is Fourier transformed, a mHzG profile will be obtained, and vice versa. In particular, the propagating field will belong to the oHzG kind when $p=0,\pm 2,\pm 4,\ldots$ and to the mHzG kind when $p=\pm 1,\pm 3,\pm 5,\ldots$. For intermediate values of p, the gHzG profiles can be regarded as a continuous transition between oHzG and mHzG profiles.

Of particular interest also is the case when the width w_0 of the initial real Gaussian apodization and the parameter a of the GRIN medium satisfy the condition $q_1=iq_1''=-ia=-ikw_0^2/2$. In this case we see from Eqs. (9) that $q_2=q_1=-ia$ remains constant for each value of z and that the wavenumber $\kappa_2=\kappa_1\times\exp(-ip\pi/2)$ rotates at a constant rate over the complex plane (κ_2',κ_2'') as the beam propagates through the GRIN medium. This situation is illustrated in Fig. 1 for a medium with $a=\sqrt{2}/\pi$ m. The input field [Fig. 1(a)] corresponds to a second-order ordinary Mathieu–Gauss (MG) beam[1] with $\kappa_1=29787\ \mathrm{m}^{-1}$ and $w_0=\sqrt{2a/k}=0.292$ mm. The 3D field distribution [Fig. 1(b)] was obtained by calculating the field with Eq. (4) at 100 transverse planes evenly spaced from the input ($p=0$) to the output ($p=2.5$) planes. Note that the width of the constituent Gaussian beams remains constant and that their propagation axes follow a sinusoidal path with period $p=4$ intersecting at even values of p. To show the case when the width of the constituent Gaussian beams is not constant, in Figs. 1(c) and 1(d) we propagate a MG profile with $\kappa_1=29{,}787\ \mathrm{m}^{-1}$ and complex $q_1=-0.490-i0.159$ m. These numerical values have been chosen such that q becomes purely imaginary at $p=0.43$.

Equation (4) can also be applied to propagate gHzG beams through complex *ABCD* systems. To illustrate this point, we show in Fig. 2 the propagation of a generalized fourth-order MG profile through two free-

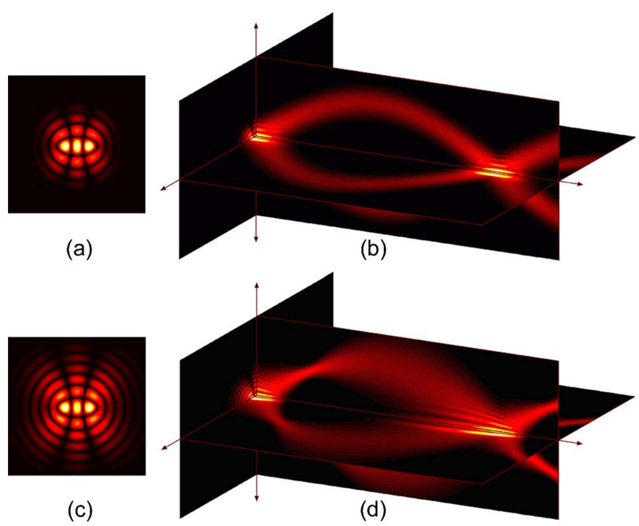

Fig. 1. (Color online) Propagation of an ordinary $\mathrm{MG}_2(\mathbf{r},\varepsilon=8)$ beam through a GRIN medium from $p=0$ to $p=2.5$. (a), (b) Constant Gaussian width. (c), (d) Variable Gaussian width.

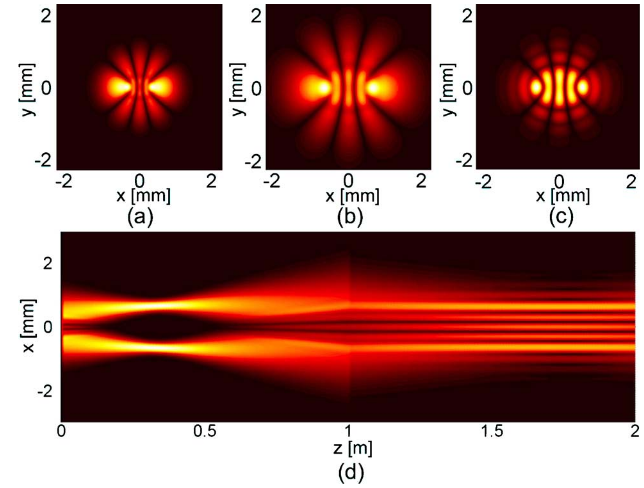

Fig. 2. (Color online) Propagation of a $\mathrm{MG}_4(\mathbf{r},\varepsilon=8)$ ($\kappa_1=-18394+i4751\ \mathrm{m}^{-1}$, $q_1=-0.331-i0.102$ m) through two free-space regions separated by a complex thin lens. Transverse field at (a) $z=0$, (b) after the complex lens $z=1$ m, (c) at $z=2$ m, (d) Evolution along the plane (z,x).

space regions of thickness 1 m separated by a thin lens with complex power $1/f=1.5-i0.03\ \mathrm{m}^{-1}$. Physically this element represents a thin lens with focal length 2/3 m apodized by a Gaussian transmittance $\exp(-0.03kr^2/2)$.

In conclusion, we have introduced a generalized form of the HzG beam whose form is invariant under paraxial optical transformations. The propagation of gHzG beams through arbitrary complex *ABCD* systems is fully characterized by the transformation of the two complex parameters (q,κ). The gHzG beams include as special cases the oHzG and mHzG beams.

This research was supported by Consejo Nacional de Ciencia y Tecnología (grant 42808) and by the Tecnológico de Monterrey (grant CAT007). J. C. Gutiérrez-Vega's e-mail address is juliocesar @itesm.mx.

References

1. J. C. Gutiérrez-Vega and M. A. Bandres, J. Opt. Soc. Am. A **22**, 289 (2005).
2. M. Guizar-Sicairos and J. C. Gutiérrez-Vega, J. Opt. Soc. Am. A **23**, 1994 (2006).
3. C. López-Mariscal, M. A. Bandres, and J. C. Gutiérrez-Vega, Opt. Eng. **45**, 068001 (2006).
4. Z. Bouchal, Czech. J. Phys. **53**, 537 (2003).
5. F. Gori, G. Guattari, and C. Padovani, Opt. Commun. **64**, 491 (1987).
6. J. C. Gutiérrez-Vega, M. D. Iturbe-Castillo, and S. Chávez-Cerda, Opt. Lett. **25**, 1493 (2000).
7. C. J. R. Sheppard and T. Wilson, IEE J. Microwaves, Opt. Acoust. **2**, 105 (1978).
8. M. Santarsiero, Opt. Commun. **132**, 1 (1996).
9. A. Belafhal and L. Dalil-Essakali, Opt. Commun. **177**, 181 (2000).
10. L. W. Casperson, D. G. Hall, and A. A. Tovar, J. Opt. Soc. Am. A **14**, 3341 (1997).
11. S. Ruschin, J. Opt. Soc. Am. A **11**, 3224 (1994).
12. A. E. Siegman, *Lasers* (University Science, 1986).
13. D. Mendlovic and H. M. Ozaktas, J. Opt. Soc. Am. A **10**, 1875 (1993).