Propagation of Helmholtz–Gauss beams in absorbing and gain media

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The propagation of Helmholtz–Gauss beams in media exhibiting loss or gain is studied. The general expressions for the field propagation, the time-averaged power on propagation, the trajectory of the beam centroid, the beam spreading, the nondiffracting distance, and the far field are derived and discussed. Explicit expressions of these parameters for Bessel–Gauss and cosine-Gauss beams are included. The general expressions can be applied straightforwardly to describe the propagation of Mathieu–Gauss and parabolic-Gauss beams in complex media as well.

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1. INTRODUCTION

In recent papers the free-space propagation of Helmholtz–Gauss (HzG) beams was theoretically[1–3] and experimentally studied.[4] The term HzG beam refers to a paraxial wave whose disturbance at the plane $z=0$ is given by the transverse field of an arbitrary nondiffracting beam modulated by a Gaussian envelope. Some special cases of the HzG beams are the known Bessel–Gauss (bg) beams[5] and the Mathieu–Gauss (mg) beams.[6] The model of the HzG beam describes in a more realistic way the propagation of ideal nondiffracting beams because HzG beams carry finite power, retain the nondiffracting propagation properties within a finite propagation distance, and can be realized experimentally to a very good approximation.[4] The properties of the HzG beams are useful for a variety of physical applications, for instance, in wireless communications, optical interconnections, laser machining, and optical tweezers.[7–10] In some cases, knowledge of the propagation characteristics of HzG beams in matter is necessary, for example, in material processing and laser damage studies,[11] modeling of semiconductors[8] and thin films,[12] and field propagation in lossy waveguides.[13]

In this paper we investigate the propagation of HzG beams in complex media having loss or gain. The general expressions for the field propagation, the time-averaged power, the beam centroid, the beam spreading, the nondiffracting distance, and the far field are obtained. Explicit expressions of these physical properties are derived and discussed for special cases, including Bessel–Gauss beams and cosine-Gauss beams, for the first time to our knowledge. Under the appropriate limit, our results reduce to the expressions reported recently by Seshadri for the propagation of a fundamental Gaussian beam in complex media.[14]

2. HELMHOLTZ–GAUSS BEAMS IN COMPLEX MEDIA

Consider a linearly polarized field $\mathbf{E}(\mathbf{r},t)=E(\mathbf{r})\times\exp(-i\omega t)\hat{\mathbf{x}}$ traveling along the positive z axis within a medium whose complex refractive index is given by

$$n=n_r+in_i. \tag{1}$$

In assuming a complex value for the refractive index, we are allowing for the possibility that the medium exhibits loss ($n_i>0$) or gain ($n_i<0$).

Let us suppose that $E(\mathbf{r})$ has a disturbance across the plane $z=0$ given by

$$E_0(\mathbf{r}_t)=\exp(-r^2/w_0^2)W(\mathbf{r}_t;k_t), \tag{2}$$

where $\mathbf{r}_t=(x,y)=(r,\phi)$ denotes the transverse coordinates, w_0 is the waist size of a Gaussian envelope, and the function $W(\mathbf{r}_t;k_t)$ satisfies the two-dimensional Helmholtz equation $(\partial_{xx}+\partial_{yy}+k_t^2)W=0$. Physically speaking, $W(\mathbf{r}_t;k_t)$ corresponds to the transverse field of an ideal nondiffracting beam $W(\mathbf{r}_t;k_t)\exp(ik_z z)$ and can be expressed as a superposition of plane waves:

$$W(\mathbf{r}_t;k_t)=\int_{-\pi}^{\pi}A(\theta)\exp[ik_t(x\cos\theta+y\sin\theta)]\mathrm{d}\theta, \tag{3}$$

where $A(\theta)$ is an arbitrary complex function and governs the oscillatory behavior of the function W in the transverse direction.

For a paraxial field traveling in the z direction, we write $E(\mathbf{r})=\Psi(\mathbf{r})\exp(ink_0 z)$, where $k_0=\omega/c$ is the free-space wavenumber and $\Psi(\mathbf{r})$ is a slowly varying complex envelope that satisfies the paraxial wave equation

$$[\partial_{xx}+\partial_{yy}+(2ink_0)\partial_z]\Psi(\mathbf{r})=0. \tag{4}$$

Localized solutions of Eq. (4) restricted to the boundary condition [Eq. (2)] can be derived by expanding the function $W(\mathbf{r}_t;k_t)$ in terms of plane waves [Eq. (3)] and solving for each constituent plane wave. This procedure has been reported in detail in Appendix A of Ref. 1 for propagation in free space ($n=1$) and can be straightforwardly extended to propagation in complex media ($n=n_r+in_i$). For briefness, we will include here only the final result:

$$E(\mathbf{r}) = \exp\left(-i\frac{k_t^2 z}{2nk_0\zeta}\right)\left[\frac{\exp(ink_0 z)}{\zeta} \times\exp\left(-\frac{r^2}{\zeta w_0^2}\right)\right] W\left(\frac{x}{\zeta},\frac{y}{\zeta};k_t\right), \tag{5}$$

where

$$\zeta(z) = 1 + i\frac{z}{nL_0} \tag{6a}$$

$$= \left(1 + \frac{n_i z}{|n|^2 L_0}\right) + i\left(\frac{n_r z}{|n|^2 L_0}\right) \tag{6b}$$

$$\equiv \zeta_r + i\zeta_i \tag{6c}$$

and $L_0 = k_0 w_0^2/2$ is the known Rayleigh distance in vacuum.

Expression (5) is a solution of the Helmholtz equation in the paraxial regime and describes the propagation of HzG beams in complex media. The beam is linearly polarized with the electric field in the x direction and the magnetic field in the y direction. Although the arguments of the function W at the plane $z=0$ are real, outside this plane they become complex, with the result that the initial shape defined by $E_0(\mathbf{r}_t)$ may change dramatically on propagation. The term in square brackets in Eq. (5) corresponds to the fundamental Gaussian beam propagating in the positive z direction.

A. Time-Averaged Poynting Vector

The z component of the time-averaged Poynting vector is given by

$$S_z = \frac{\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*)}{2}\cdot\hat{\mathbf{z}} = \frac{\epsilon_0 c n_r}{2}|E|^2, \tag{7}$$

where ϵ_0 is the free-space permittivity, the superscript (*) denotes complex conjugation, and the operator $\mathrm{Re}(\cdot)$ retrieves the real part of the argument.

The total time-averaged power $P(z)$ of a HzG beam depends on the propagation distance z and is obtained by substituting Eq. (5) into Eq. (7) and integrating over the transverse plane, namely, $P(z)=\int\int S_z \mathrm{d}A$, where $\mathrm{d}A=\mathrm{d}x\mathrm{d}y$ is the differential element of area. We obtain

$$P(z) = \frac{n_r\epsilon_0 c\,\exp(-2k_0 z n_i)}{2|\zeta|^2}\exp\left[-\frac{k_t^2 z}{k_0|n|^2|\zeta|^2}\left(n_i + \frac{z}{L_0}\right)\right] \times\int\int\exp\left(-\frac{2\zeta_r}{|\zeta|^2 w_0^2}r^2\right)\left|W\left(\frac{x}{\zeta},\frac{y}{\zeta};k_t\right)\right|^2 \mathrm{d}A. \tag{8}$$

It can be easily shown that Eq. (8) reduces to a constant value when $n_i=0$.

To have a converging integral in Eq. (8), the condition

$$\zeta_r = 1 + \frac{n_i z}{|n|^2 L_0} > 0 \tag{9}$$

needs to be satisfied. As we will see later, this condition imposes a limiting propagation distance in the positive z direction for gain media ($n_i<0$), namely,

$$z_{cr} = \frac{|n|^2 L_0}{|n_i|}. \tag{10}$$

B. Special Cases

Equation (8) provides the power transported in the z direction of a HzG beam propagating in a medium with complex refraction index. There are, of course, an infinite number of possible choices for the function W but of particular interest are the fundamental and orthogonal families of HzG beams[1] expressed in Cartesian (cosine-Gauss beams), circular (Bessel–Gauss beams), elliptic (Mathieu–Gauss beams),[6] and parabolic (parabolic-Gauss beams)[15] coordinates.

For comparison purposes we will work with the normalized beam power, given by

$$\bar{P}(z) = P(z)/P(0), \tag{11}$$

such that $\bar{P}(0)=1$.

1. Tilted Plane-Wave-Gaussian beam

The simplest form of a HzG beam corresponds to a simple tilted plane-wave-Gaussian beam for which $W=\exp(ik_t x)$. From Eq. (5) we have explicitly

$$E_{PW}(\mathbf{r}) = \exp\left(-i\frac{k_t^2 z}{2nk_0\zeta}\right)\left[\frac{\exp(ink_0 z)}{\zeta} \times\exp\left(-\frac{r^2}{\zeta w_0^2}\right)\right]\exp\left(ik_t\frac{x}{\zeta}\right). \tag{12}$$

Inserting Eq. (12) into Eq. (8) yields the normalized power of the beam at a propagation distance z:

$$\bar{P}_{PW}(z) = f(z)\exp\left[2\gamma^2\frac{\zeta_i^2}{\zeta_r|\zeta|^2}\right], \tag{13}$$

where $\gamma \equiv k_t w_0/2$ and

$$f(z) \equiv \frac{1}{\zeta_r}\exp\left[-\frac{k_t^2 z}{k_0|n|^2|\zeta|^2}\left(n_i + \frac{z}{L_0}\right)\right]\exp(-2k_0 z n_i). \tag{14}$$

The time-averaged power of the fundamental Gaussian beam is recovered in the case $k_t=0$ for which $W=1$. From Eq. (13),

$$\bar{P}_{GB}(z) = \frac{1}{\zeta_r}\exp(-2k_0 z n_i), \tag{15}$$

is obtained, which is indeed the expression already derived by Seshadri in Ref. 14.

2. Cosine-Gauss Beams

The cosine-Gauss beams correspond to $W=\cos(k_t x)$ resulting from the superposition of two ideal plane waves, $\exp(ik_t x)/2+\exp(-ik_t x)/2$. By inserting W into Eq. (8) and performing the integral over the transverse plane, we determine the following power distribution:

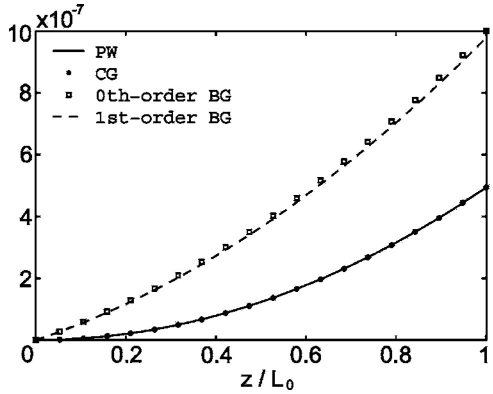

Fig. 1. Plot of $\bar{P}(z)/f(z)-1$ for the tilted plane-wave-Gaussian beam, the cosine-Gauss beam, and the zeroth- and first-order Bessel–Gauss beams.

$$\bar{P}_{CG}(z) = f(z)\frac{[1+\exp(2\gamma^2/\zeta_r)]}{[1+\exp(2\gamma^2)]}\exp\left(2\gamma^2\left[1-\frac{\zeta_r}{|\zeta|^2}\right]\right). \tag{16}$$

3. Bessel–Gauss Beams

The well-known Bessel–Gauss beams[1,5] are the fundamental family of HzG beams in circular coordinates with $W=J_m(k_t r)\exp(im\phi)$. By inserting W in Eq. (8) and performing the integral over the transverse plane, we determine, for the first time to our knowledge, the power evolution of the mth-order Bessel–Gauss beams in complex media:

$$\bar{P}_{BG}(z) = f(z)\exp\left(-\gamma^2\left[\frac{\zeta_r^2-\zeta_i^2}{|\zeta|^2\zeta_r}-1\right]\right)\frac{I_m(\gamma^2/\zeta_r)}{I_m(\gamma^2)}, \tag{17}$$

where $I_m(\cdot)$ is the mth-order modified Bessel function of the first kind.

Although power expressions in Eqs. (13), (16), and (17) are different, we have found that on numerical evaluation they have very close values, which indicates that the main mechanism of power loss is driven by the factor $f(z)$. Close examination reveals that, for typical operating parameters, the lowest loss beam is actually the zeroth-order Bessel–Gauss beam and the highest loss occurs with the cosine-Gauss beam, as shown in Fig. 1.

3. ANALYSIS OF THE STATISTICAL PROPAGATION PARAMETERS

Equation (5) describes the propagation of a HzG beam in a complex medium. To gain basic understanding of the propagation features, we note that a HzG beam is formed as a superposition of tilted plane-wave-Gaussian beams [Eq. (12)] that have their foci coincident with the plane $z=0$, whose mean propagation axes lie on the surface of a cone, and whose amplitudes are modulated angularly by the function $A(\theta)$ [Eq. (3)]. Basic information on the propagation of the HzG beams can be obtained by analyzing the average properties of this single tilted plane-wave-Gaussian beam.

A. Centroid of the Constituent Tilted Plane-Wave-Gaussian Beam

The first-order moment (i.e., the expected value of x),[16]

$$\langle x\rangle = \frac{1}{P(z)}\int_A xS_z\mathrm{d}A, \tag{18}$$

gives the centroid of the beam. Inserting Eq. (12) into Eqs. (7) and (8) and performing the integral in Eq. (18), we find the behavior of the beam centroid on propagation as a function of z, namely,

$$\langle x\rangle = \frac{k_t n_r}{k_0|n|^2}\frac{z}{1+z(n_i|n|^{-2}L_0^{-1})}, \tag{19}$$

where the condition $\zeta_r>0$ must be fulfilled to have a converging integral in Eq. (8). Just as expected,[1] if we set $n_i=0$, the beam centroid propagates in a straight line with slope $k_t/(k_0 n_r)$; see Fig. 2(a).

For absorbing media ($n_i>0$) note that, unlike the propagation in purely real index media, the propagation of the beam centroid is not a straight line but rather a curve that tends asymptotically to the constant value

$$\langle x\rangle_{z\to\infty} = r_C = \frac{k_t w_0^2}{2}\frac{n_r}{n_i}. \tag{20}$$

At large propagation distances, this effect leads the HzG beam to be formed by a superposition of constituent Gaussian envelopes having mean value wave vectors parallel to the longitudinal z direction, but whose centroids are on a circumference of radius r_C around the z axis; see Fig. 2(b).

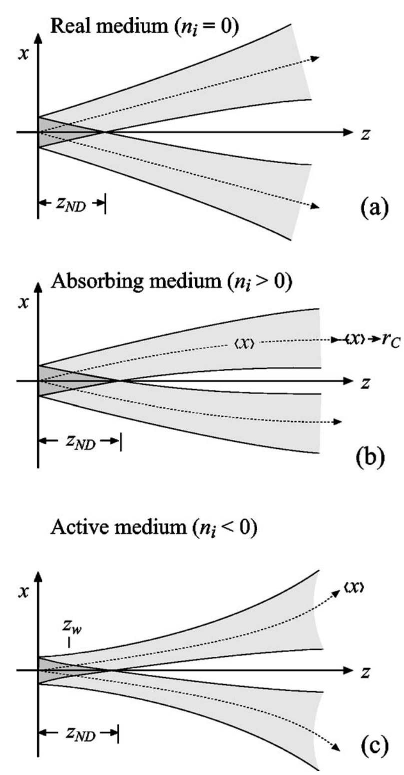

Fig. 2. Behavior of the tilted plane-wave-Gaussian beam components of the HzG beams in (a) medium with purely real refraction index, (b) absorbing medium, and (c) gain medium.

For gain media ($n_i<0$), the beam centroid $\langle x\rangle$ of the constituent tilted plane-wave-Gaussian beams gets away from the z axis and tends to infinity when z tends to $z_{cr}=|n|^2 L_0/|n_i|$ (i.e., when $\zeta_r=0$); see Fig. 2(c).

B. Physical Discussion of the Behavior of the Beam Centroid

A physical discussion of the nonlinear trajectory of the beam centroid described by Eq. (19) is in order. To the reader, this seems to imply that the medium exerts some sort of force that curves the beams toward the axis in the case of absorption or away from the axis in the case of gain. It is tempting to visualize the tilted plane-wave-Gaussian beam as simply a fundamental Gaussian beam propagating at a small angle with respect to the z axis. Since the medium is homogeneous and isotropic, then a Gaussian beam traveling in any direction should be simply a rotated version of a Gaussian beam of the same width traveling in a different direction, and both should be symmetric about their straight axes. This, however, is not what Eq. (19) and Figs. 2(b) and 2(c) show.

The nonlinear trajectory of the beam centroid on propagation is explained by noting from Eq. (12) that the tilted plane-wave-Gaussian beam at the plane $z=0$ reduces to $E_0(x,y)=\exp(-r^2/w_0^2)\exp(ik_t x)$. While the plane wave is tilted an angle $\arctan(k_t/k)$ about the z axis, the Gaussian modulation remains over the plane (x,y). Evidently, $E_0(x,y)$ does not correspond strictly to the field at $z=0$ of a tilted fundamental Gaussian beam, even when the tilt angle is small. This lack of symmetry between the amplitude and the phase distribution across the plane $z=0$ produces the result that the propagation of the beam centroid of a tilted plane-wave-Gaussian beam is not a straight line as occurs for a tilted Gaussian beam. Although this slight difference does not have important consequences in free-space propagation, for absorbing and gain media it cannot be neglected and is crucial in the physical interpretation of the propagation characteristics of the HzG beams.

It is instructive to show that the position of the beam centroid $\langle x\rangle$ described by Eq. (19) indeed corresponds to the position of the maximum of the beam intensity along the transverse coordinate x. From basic calculus, we have

$$\left.\frac{\mathrm{d}|E_{PW}(x,z)|}{\mathrm{d}x}\right|_{x=x_{max}}=0, \tag{21}$$

where $|E_{PW}|$ is the amplitude of the tilted plane-wave-Gaussian beam. Replacing Eq. (12) and solving for x_{max}, it is straightforward to prove that x_{max} is given exactly by Eq. (19).

The computed trajectory of a beam centroid is dependent on the selection of the plane on which the integration is performed. Thus if a different selection of integration plane is made, this trajectory will be modified; this effect is particularly significant in absorptive and gain media. However, it is important to note that the nonlinear trajectory of the beam centroid is predominantly a result of the lack of symmetry between the amplitude and the phase distribution and not a mere effect of the choice of integration plane. To better visualize this, the reader is referred to Fig. 3, a display of constant amplitude contours of the plane-wave-Gaussian beam in absorptive media; the dashed curve depicts the curve given by Eq. (19). It is evident that, although the trajectory of the maximum intensity of the beam would be modified on a rotation of the integration plane, this trajectory will remain nonlinear.

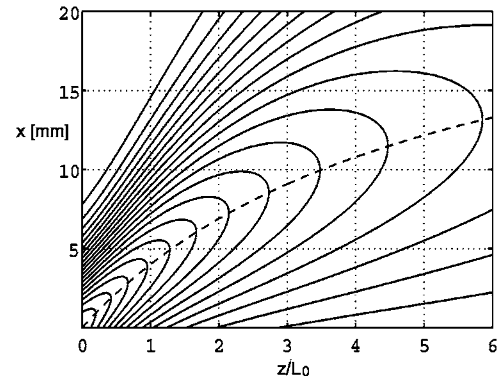

Fig. 3. Constant amplitude contours of the plane-wave-Gaussian beam in absorptive media; dashed curve depicts the curve given by Eq. (19).

As pointed out in Subsection 3.A, for gain media ($n_i<0$), the beam centroid $\langle x\rangle$ increases indefinitely and tends to infinity at $z=z_{cr}$. Evidently this mathematical result is counterintuitive and should be physically interpreted. First, we need to consider that in any physical gain media the gain will present saturation effects that are not accounted for by the constant complex refractive index, since at every point of the gain media there is only a finite amount of energy available for field amplification. The approximation of gain through the complex index of refraction is limited to a finite distance (presumably smaller than z_{cr}), where the gain saturation effects can be neglected. Second, once the beam centroid gets quickly away from the z axis, the application of the paraxial approximation is no longer valid. This divergence of the beam centroid will not happen in a rigorous analysis of the nonparaxial propagation of an initially Gaussian field in a gain medium, even if the medium can give an infinite amount of power to the beam.

Finally, note that the initial slope (at $z=0$) of the beam centroid is always decreased in gain or absorbing media by a factor of $n_r^2/|n|^2$, regardless of the sign of n_i. This effect will contribute to the increase of the nondiffracting distance of the HzG beams. In Fig. 4 we show the field amplitude of a cosine-Gauss beam along the plane (x,z) propagating within a lossy medium. The plot was obtained directly from Eq. (5) with $W=\cos(k_t x)$, and the beam maximum at each plane z was normalized to unity for visualization purposes.

C. Beam Width

We associate the beam waist to the variance of the beam intensity by[16]

$$w^2(z)=4(\langle x^2\rangle-\langle x\rangle^2) \tag{22}$$

$$=\frac{w_0^2|\zeta|^2}{\zeta_r}, \qquad \zeta_r>0, \tag{23}$$

where we see that the beam waist does not have the usual quadratic dependence that regular propagation of HzG

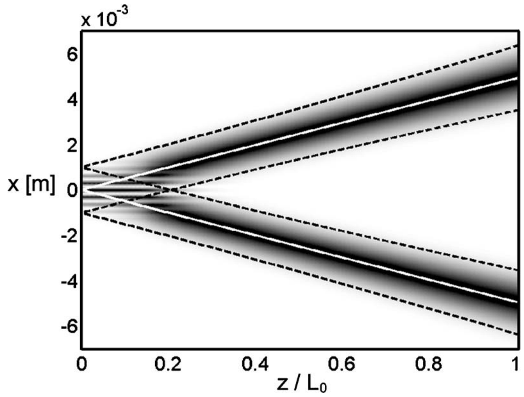

Fig. 4. Propagation along the plane (x,z) of a cosine-Gauss beam with parameters $\lambda=632.8$ nm, $k_0=2\pi/\lambda$, $k_t=0.001k_0$, within a lossy medium with refraction index $n=1+i0.01$. For visualization purposes the field amplitude is normalized at each plane z.

beams exhibits. It is also important to note that the waist of the propagated beam does not depend on k_t; that is, the waist growth is exactly the same as for a fundamental Gaussian beam in complex media.

The new Raleigh distance of the beam, defined as the distance at which the mean-square width becomes twice the value at $z=0$, is given by $L=L_0|n|$. This increase of the Raleigh distance denotes that the spreading of the beam is decreased, at least up to that point. Notice that the distance L reduces to the expected value $L_0 n_r$ when dealing with purely real refractive index.

For gain media, the beam waist is shifted forward, is actually slightly smaller than that of $z=0$, and occurs at a positive value of z given by [see Fig. 2(c)]

$$z_w \approx |n_i| L_0/2, \tag{24}$$

where the approximation $|n_i| \ll |n|$ has been applied. At $z=z_w$ the waist takes the minimum value

$$w^2(z_w) \approx w_0^2\left(1-\frac{n_i^2}{4|n|^2}\right). \tag{25}$$

As z is increased beyond z_w, the width increases monotonically up to $z=z_{cr}$, where the beam waist goes to an infinite value. This fact led Seshadri[14] to state that the fundamental Gaussian beam in gain media "spreads rapidly and disintegrates." A physical interpretation of this mathematical result has been provided in Subsection 3.B.

4. ANALYSIS OF THE NONDIFFRACTING DISTANCE

By analyzing the statistical moments of the tilted plane-wave-Gaussian beam, we can estimate the nondiffracting distance of the HzG beam in complex media. This distance is found when the beam width is equal to the distance of the beam centroid from the propagation axis and delimits the region where significant interference of the constituent tilted plane-wave-Gaussian beams occurs; since the medium is linear, two or more constituent waves with the same transverse wavenumber will have significant interference before this critical point.

The nondiffracting distance z_{ND} can be straightforwardly obtained for $n_i=0$ and is given by[1]

$$z_{ND} = \frac{n_r}{\sqrt{\gamma^2-1}} L_0. \tag{26}$$

This expression indicates that the zero crossing of the waist occurs in the same value for positive and negative propagation distances z. This result does not hold for absorbing or gain media.

To estimate the nondiffracting distance of the HzG beam, we must obtain the values of z that satisfy $\langle x\rangle^2 = w^2(z)$; then

$$\frac{k_t^2 n_r^2}{k_0^2|n|^4}\frac{z^2}{\left(1+\frac{n_i z}{|n|^2 L_0}\right)^2} = \frac{w_0^2|\zeta|^2}{1+\frac{n_i z}{|n|^2 L_0}}. \tag{27}$$

Rearranging, we obtain the cubic equation

$$z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0, \tag{28}$$

where

$$\alpha_2 \equiv \frac{L_0}{n_i}(|n|^2 + 2n_i^2 - \gamma^2 n_r^2), \tag{29a}$$

$$\alpha_1 \equiv 3|n|^2 L_0^2, \tag{29b}$$

$$\alpha_0 \equiv \frac{|n|^4 L_0^3}{n_i}. \tag{29c}$$

A. Remarks on the Discriminant

Since all coefficients of the cubic equation are real, at least one solution should exist among the real numbers. We can use the sign of the discriminant of Eq. (28) to study its solutions. The discriminant is given by

$$\Delta = 4\alpha_1^3 - \alpha_1^2\alpha_2^2 + 4\alpha_0\alpha_2^3 - 18\alpha_0\alpha_1\alpha_2 + 27\alpha_0^2. \tag{30}$$

We analyzed the order of magnitude of the additive terms that constitute the discriminant, assuming typical physical parameters ($\lambda_0 \sim 10^{-6}$ m, $n_r \sim 1$, $n_i \sim 10^{-2}$, $w_0 \sim 10^{-3}$ m), and found that $4\alpha_0\alpha_2^3$ is about 5 orders of magnitude larger than the rest of the terms. This means that the discriminant is mainly determined by

$$\Delta \approx 4\alpha_0\alpha_2^3 = \frac{|n|^4 L_0^6}{n_i^4}(|n|^2 + 2n_i^2 - \gamma^2 n_r^2)^3, \tag{31}$$

as long as

$$||n|^2 + 2n_i^2 - \gamma^2 n_r^2| > 10^{-1}, \tag{32}$$

and that we can estimate the sign of the determinant by

$$\mathrm{sign}(\Delta) = \mathrm{sign}(|n|^2 + 2n_i^2 - \gamma^2 n_r^2). \tag{33}$$

Care should be taken when using the approximation in Eq. (33); if the inequality in expression (32) is very close to zero, the other terms of the discriminant should be considered.

If $\Delta < 0$, there are three real solutions of Eq. (28); in gain media, only one of these solutions is positive and defines the nondiffracting distance of the beam. However, in absorbing media two of these solutions are positive, defining along with the nondiffracting distance a second propagation distance where the constituent tilted plane-wave-Gaussian beams will interfere again. This is evident if we consider the fact that the width is a monotonically increasing function while the beam centroid tends asymptotically to a constant value. However, when this interference occurs again, at very large values of z the beam no longer exhibits the transverse pattern of an ideal nondiffracting beam but rather a simple superposition of expanding Gaussian amplitude envelopes.

On the other hand, if $\Delta > 0$, then the equation has only one real root that is positive in gain media and negative in absorbing media. This real solution always appears outside the valid range for waist and beam centroid computations ($\zeta_r < 0$) and thus is not a valid solution. This effect of no zero crossings of the waist occurs when the transverse wavenumber k_t is smaller than a certain limit so that the beam centroid distance can never exceed the waist growth.

We can see this condition as an approximate lower limit for the γ parameter that is required for a well-defined nondiffracting distance to appear, i.e., a zero crossing of the waist on the z axis,

$$\gamma^2 > \frac{|n|^2 + 2n_i^2}{n_r^2}. \tag{34}$$

Notice that expression (34) reduces to the condition

$$\gamma^2 > 1, \tag{35}$$

if $n_i = 0$ as required for real solutions of Eq. (26) in lossless media.[1]

B. Exact Analytic Solution

The cubic equation (28) was solved analytically with the classical Cardan method.[17] The general solution is given by

$$z = \frac{p}{3u} - u - \frac{\alpha_2}{3}, \tag{36}$$

where the three values of z are given by the three distinct values of u,

$$u = \exp\left(i\frac{\pi m}{3}\right)\left(\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3}, \qquad m = 0,1,2, \tag{37a}$$

and where we have defined

$$p = \alpha_1 - \frac{\alpha_2^2}{3}, \tag{37b}$$

$$q = \alpha_0 + \frac{2\alpha_2^3 - 9\alpha_2\alpha_1}{27}. \tag{37c}$$

This is a general solution of the cubic equation with just one restriction; in the unlikely case where $p = q = 0$, the solution is the triple real root $z = -\alpha_2/3$.

1. Zero-Order Approximation

Since expression (36) is algebraically complicated, it forbids direct inspection of the mechanisms that modify the nondiffracting distance of the HzG beams due to the complex refraction index. For a zero-order approximation, if we neglect the waist dependence of the z coordinate and assume that the beam centroid propagates in a straight line, we obtain a rough approximation of the nondiffracting distance:

$$z_{ND} = \frac{w_0 k_0 |n|^2}{n_r k_t}. \tag{38}$$

The nondiffracting distance is increased for both gain and passive media by a factor of $|n|^2/n_r^2$ with respect to real refractive index media, as is expected from the corresponding decrease in initial propagation angle that was discussed before. Notice that if we set $n_i = 0$, we obtain

$$z_{ND} = \frac{w_0 k_0 n_r}{k_t}, \tag{39}$$

which is exactly the expression of nondiffracting distance that is obtained from Eq. (26) by neglecting the waist axial dependence.[1]

2. First-Order Approximation

In this approximation we partially include the effects of propagation curve and waist evolution when computing the nondiffracting distance of the HzG beam. From Eq. (28) we consider only the most significant coefficients, neglecting the cubic dependence and assuming $\alpha_1 = 0$. In this case the expression

$$z_{ND} = \sqrt{-\frac{\alpha_0}{\alpha_2}} = \frac{|n|^2}{n_r\sqrt{\gamma^2 - (2n_i^2 + |n|^2)/n_r^2}} L_0 \tag{40}$$

gives an approximation of the nondiffracting distance of the beam. Notice that if $n_i = 0$ we exactly obtain Eq. (26) and that the condition in Eq. (40) to have zero crossings of the waist is exactly the one that was estimated from the discriminant expression (34).

Equation (40) reveals two distinct mechanisms by which the imaginary part of the refractive index contributes to the increase of the nondiffracting distance. First, the decrease in the initial angle of propagation, as was analyzed in the zero-order approximation, appears in this approximation (the $|n|^2/n_r$ factor), contributing to the increase of the nondiffracting distance. The second mechanism of increase of the nondiffracting distance is evident if we rewrite Eq. (40) as

$$z_{ND} = \sqrt{-\frac{\alpha_0}{\alpha_2}} = \frac{|n|^2}{n_r\sqrt{\gamma^2 - 1 - 3n_i^2/n_r^2}} L_0 \tag{41}$$

and see that, assuming that the condition in expression (34) is fulfilled and for fixed parameters k_t and w_0, the nondiffracting distance will increase owing to the ratio n_i^2/n_r^2 regardless of the sign of n_i.

On numerical evaluation of the different approximations for the nondiffracting distance, we have observed that Eq. (36) returns values that are even higher than those obtained from the zero- and first-order approximations, thus leading us to think that the mechanisms that were excluded in the approximations actually contribute to further increase the nondiffracting distance of the HzG beam.

5. FAR FIELD OF THE HELMHOLTZ–GAUSSIAN BEAMS

The far-field transverse beam profile of a HzG field in a complex medium can be examined simply by setting $z \gg L_0$. In this approximation we let

$$\zeta(z) \approx i\frac{z}{nL_0}. \tag{42}$$

We introduce this approximation in Eq. (5) along with the transverse pattern of the HzG beam:

$$W\left(\frac{x}{\zeta},\frac{y}{\zeta};k_t\right) = \int_{-\pi}^{\pi} A(\theta)\exp\left[i\frac{k_t}{\zeta}(x\cos\theta + y\sin\theta)\right]\mathrm{d}\theta. \tag{43}$$

Little information can be retrieved from the direct substitution of the approximation in expression (42). After some algebraic manipulation we reach the expression

$$E_x = \frac{L_0 n}{iz}\exp(-\gamma^2)\exp(ik_0nz)\exp\left(\frac{ik_0n_rr^2}{2z}\right)\exp\left(\frac{L_0\gamma^2n_r^2}{zn_i}\right) \times \int_{-\pi}^{\pi} A(\theta)\exp\left\{-\frac{n_ik_0}{2z}[(x-r_C\cos\theta)^2 + (y-r_C\sin\theta)^2]\right\}\mathrm{d}\theta. \tag{44}$$

Notice that the integral is only convergent for absorbing media and that, for that case, the centroid of each tilted plane-wave-Gaussian component is located at a cylindrical radial distance of $r_C = k_t n_r w_0^2/2n_i$ and at an azimuthal angle θ. This agrees with the beam centroid prediction for large values of z given in Eq. (20) that was computed for $\theta = 0$.

Also notice that the integral sums completely real Gaussian contributions that are modulated by the term $A(\theta)$. This superposition will not have the nondiffracting characteristics of the originally constituent tilted plane waves.

6. CONCLUSIONS

We studied the time-averaged power, the beam centroid, the beam spreading, the nondiffracting distance, and the far field of the HzG beams propagating in complex media. In the course of obtaining these expressions, we derived the full spatial evolution of the HzG beams in media exhibiting loss or gain. Explicit expressions of these properties were provided for cosine-Gauss beams and Bessel–Gauss beams for the first time to our knowledge. The general expressions can be applied straightforwardly to describe the propagation of Mathieu–Gauss and parabolic-Gauss beams in complex media as well.

The analysis revealed that the spreading of the constituent tilted plane-wave-Gaussian waves forming the HzG beam is reduced, leading to an increase of the nondiffracting distance with respect to the propagation in a medium with purely real index of refraction. The trajectory of the beam centroid of the constituent waves is not a straight line but rather a curve that either tends asymptotically to a constant value for absorbing media or diverges for gain media. This effect in absorbing media leads the HzG beam to be formed by a superposition of Gaussian envelopes, having mean value wave vectors parallel to the longitudinal z direction, but whose centroids are on a circumference of radius r_C around the z axis. We also found that, for gain media, the beam waist of each constituent wave occurs at a positive value of $z = z_w$ and is actually smaller than that of $z = 0$.

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