Vector Helmholtz–Gauss and vector Laplace–Gauss beams

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We demonstrate the existence of vector Helmholtz–Gauss (vHzG) and vector Laplace–Gauss beams that constitute two general families of localized vector beam solutions of the Maxwell equations in the paraxial approximation. The electromagnetic components are determined starting from the scalar solutions of the two-dimensional Helmholtz and Laplace equations, respectively. Special cases of the vHzG beams are TE and TM Gaussian vector beams, nondiffracting vector Bessel beams, polarized Bessel–Gauss beams, modes in cylindrical waveguides and cavities, and scalar Helmholtz–Gauss beams. The general expression of the vHzG beams can be used straightforwardly to obtain vector Mathieu–Gauss and vector parabolic-Gauss beams, which to our knowledge have not yet been reported. © 2005 Optical Society of America

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Laser beams are commonly studied within the framework of the scalar and paraxial approximations of the Helmholtz equation. Hermite–Gauss, Laguerre–Gauss, and Ince–Gauss beams[1] constitute the three fundamental and orthogonal families of solutions of the paraxial wave equation (PWE). Localized scalar beam solutions have been also constructed that involve the product of a Gaussian beam and complex scaled solutions of the two-dimensional (2D) Helmholtz and Laplace equations.[2–4] For most applications that do not involve the polarization properties of beams, the scalar framework is adequate. However, when the polarization of the field is of major concern, knowledge of the vector beam solutions is essential. In this direction, the problem of finding vector solutions of Maxwell equations has been studied.[5–8]

In this Letter we introduce two general families of localized vector beam solutions of the Maxwell equations in the paraxial regime. The first family of solutions is constructed starting from the scalar solutions of the 2D Helmholtz equation; thus we refer to them as vector Helmholtz–Gauss (vHzG) beams. The transverse fields appear naturally as solutions of the vector PWE by applying the separation-of-variables method. Starting from the scalar solutions of the 2D Laplace equation, we also determine a second family of localized beams that we refer to as vector Laplace–Gauss (vLpG) beams. We follow a coordinate-free approach rather than proposing solutions for the vector PWE in a particular coordinate system.[8]

The connection with results published elsewhere[5–8] is made by noting that, under the appropriate limits, the vHzG beams reduce to the special cases of scalar Helmholtz–Gauss beams,[2–4] TE and TM Gaussian vector beams,[6] nondiffracting vector Bessel beams,[7] vector Bessel–Gauss beams,[8] and propagating modes supported by waveguides and cavities with constant cross section.[9] The familiar linearly polarized Gaussian beam is found to be a special case of the vLpG beams. Additionally, the general expressions of the vHzG beams can be applied to determine the vector Mathieu–Gauss and vector parabolic-Gauss beams, which to our knowledge have not yet been derived in the literature. The conditions for the validity of the paraxial approximation are discussed.

Consider the free-space propagation of a monochromatic electromagnetic beam along the positive z axis of a coordinate system $\mathbf{r}=(\mathbf{r}_t,z)$, where $\mathbf{r}_t=(\hat{\mathbf{x}}x+\hat{\mathbf{y}}y)$ is the transverse radius vector. The electric and magnetic fields are written as $\mathbf{E}=(\mathbf{E}_t+\hat{\mathbf{z}}E_z)\exp(ikz)$ and $\mathbf{H}=(\mathbf{H}_t+\hat{\mathbf{z}}H_z)\exp(ikz)$, where $k=\omega(\mu_0\epsilon_0)^{1/2}$ is the wave number and the subscripts t and z stand for transverse and longitudinal components, respectively. From the perturbative series expansion of Maxwell equations provided by Lax *et al.*,[5] it is known that zeroth-order fields are purely transverse and satisfy the vector PWE

$$[\nabla_t^2+2ik\partial/\partial z]\{\mathbf{E}_t,\ \mathbf{H}_t\}=0, \tag{1}$$

where $\nabla_t=\hat{\mathbf{x}}\partial/\partial x+\hat{\mathbf{y}}\partial/\partial y$ is the transverse nabla operator. The Lax expansion also showed that in next-order correction a small longitudinal field component must be present, and its value is obtained from the transverse components through

$$\{E_z,\ H_z\}=(i/k)\nabla_t\cdot\{\mathbf{E}_t,\ \mathbf{H}_t\}. \tag{2}$$

Additionally, to be consistent with Maxwell's equations, the transverse fields $\mathbf{E}_t$ and $\mathbf{H}_t$ and unit vector $\hat{\mathbf{z}}$ are mutually perpendicular and satisfy

$$\mathbf{H}_t=(\epsilon_0/\mu_0)^{1/2}\hat{\mathbf{z}}\times\mathbf{E}_t. \tag{3}$$

To obtain a rigorous analytical solution to Eq. (1), we first note that Gaussian localization is essential to propagation in free space; then we write $\mathbf{E}_t(\mathbf{r})$ as

$$\mathbf{E}_t(\mathbf{r})=\mathbf{U}(X,Y,\zeta)G(\mathbf{r}), \tag{4}$$

where $(X,Y)=(x/\zeta,y/\zeta)$ are scaled Cartesian coordinates, $\zeta(z)=1+iz/z_R$, and

$$G(\mathbf{r})=\zeta^{-1}\exp(-r^2/w_0^2\zeta) \tag{5}$$

is the familiar Gaussian beam with waist size w_0 and Rayleigh range $z_R=kw_0^2/2$.

Inserting Eq. (4) into Eq. (1) and noting that $G(\mathbf{r})$ satisfies the PWE produces the equation for $\mathbf{U}$:

$$\nabla_T^2\mathbf{U}-(4\zeta^2/w_0^2)\partial\mathbf{U}/\partial\zeta=0, \tag{6}$$

where $\nabla_T = \hat{\mathbf{x}}\partial/\partial X + \hat{\mathbf{y}}\partial/\partial Y$ is the transverse nabla in the scaled coordinates. Equation (6) admits the separation of variables $\mathbf{U} = \boldsymbol{\Psi}(X,Y)Z(\zeta)$, following which we can easily find that

$$Z(\zeta) = \exp[k_t^2 w_0^2(\zeta^{-1} - 1)/4], \tag{7}$$

where k_t^2 is the separation constant and $\boldsymbol{\Psi}(X,Y)$ satisfies the 2D vector Helmholtz equation

$$\nabla_T^2 \boldsymbol{\Psi} + k_t^2 \boldsymbol{\Psi} = 0. \tag{8}$$

Vector solutions of the three-dimensional Helmholtz equation were studied in Refs. 9 and 10. Following the same approach, we found that Eq. (8) admits two independent vector solutions of the form

$$\boldsymbol{\Psi}^{(1)} = \nabla_T W(X,Y), \quad \boldsymbol{\Psi}^{(2)} = -\hat{\mathbf{z}} \times \boldsymbol{\Psi}^{(1)}, \tag{9}$$

where $W(X,Y)$ is a solution of the 2D scalar Helmholtz equation $\nabla_T^2 W + k_t^2 W = 0$. The physical meaning of separation constant k_t is now clear, as it governs the oscillatory behavior of the function W in the transverse direction.

Substituting Eqs. (9) into Eq. (4) and using Eqs. (1)–(3), we obtain for the first-class vector beam solution

$$\mathbf{E}_t^{(1)} = Z(\zeta)G(\mathbf{r})\nabla_T W, \tag{10a}$$

$$E_z^{(1)} = -\frac{iZ(\zeta)G(\mathbf{r})}{\zeta} \times \left(\frac{k_t^2}{k} W + \frac{2}{kw_0}\nabla_T W \cdot \frac{\mathbf{r}_t}{w_0}\right), \tag{10b}$$

$$\mathbf{H}_t^{(1)} = \sqrt{\epsilon_0/\mu_0}\, Z(\zeta)G(\mathbf{r})(\hat{\mathbf{z}} \times \nabla_T W), \tag{10c}$$

$$H_z^{(1)} = -\sqrt{\epsilon_0/\mu_0}\,\frac{2i}{kw_0}\frac{Z(\zeta)G(\mathbf{r})}{\zeta} \times (\hat{\mathbf{z}} \times \nabla_T W) \cdot \frac{\mathbf{r}_t}{w_0}. \tag{10d}$$

In the same way, the second-class vector beam solution is given by

$$\mathbf{E}_t^{(2)} = -Z(\zeta)G(\mathbf{r})(\hat{\mathbf{z}} \times \nabla_T W), \tag{11a}$$

$$E_z^{(2)} = \frac{2i}{kw_0}\frac{Z(\zeta)G(\mathbf{r})}{\zeta}(\hat{\mathbf{z}} \times \nabla_T W) \cdot \frac{\mathbf{r}_t}{w_0}, \tag{11b}$$

$$\mathbf{H}_t^{(2)} = \sqrt{\epsilon_0/\mu_0}\, Z(\zeta)G(\mathbf{r})\nabla_T W, \tag{11c}$$

$$H_z^{(2)} = -\sqrt{\epsilon_0/\mu_0}\,\frac{iZ(\zeta)G(\mathbf{r})}{\zeta} \times \left(\frac{k_t^2}{k} W + \frac{2}{kw_0}\nabla_T W \cdot \frac{\mathbf{r}_t}{w_0}\right). \tag{11d}$$

A general vector beam solution is expressed by the superposition $\mathbf{E} = \alpha \mathbf{E}^{(1)} + \beta \mathbf{E}^{(2)}$, where α and β are constants. Note that solution (11) can also be obtained from solution (10) applying the duality property, i.e., replacing $\mathbf{E}$ with $(\mu_0/\epsilon_0)^{1/2}\mathbf{H}$ and $(\mu_0/\epsilon_0)^{1/2}\mathbf{H}$ with $-\mathbf{E}$.

We have demonstrated, then, that localized vector beam solutions of the Maxwell equations can be constructed starting from the scalar solutions of the earlier 2D Helmholtz equation. These scalar solutions can formally be written as $W(X,Y) = \int_{-\pi}^{\pi} A(\theta)\exp[ik_t(X\cos\theta + Y\sin\theta)]\mathrm{d}\theta$, where $A(\theta)$ is an arbitrary complex function. There are, of course, an infinite number of possible solutions, but of particular interest are the fundamental and orthogonal families of eigenfunctions of the 2D Helmholtz equation expressed in Cartesian, circular, elliptic, and parabolic coordinates. The case of the circular coordinates (r,ϕ) corresponds to eigenfunctions $W(x,y) = J_m(k_t r)\exp(\pm im\phi)$, for which vHzG beams reduce to the mth-order vector Bessel–Gauss beams studied previously by Hall.[8] For elliptic coordinates (ξ,η), vector-even Mathieu–Gauss beams of mth-order can be constructed from the eigenfunctions $\mathrm{Je}_m(\xi,q)\mathrm{ce}_m(\eta,q)$, where $\mathrm{Je}(\cdot)$ and $\mathrm{ce}(\cdot)$ are the radial and angular even Mathieu functions of mth-order and parameter q, respectively.[11] For parabolic coordinates (u,v), vector-even parabolic-Gauss beams can be constructed from the eigenfunctions $\mathrm{Pe}(u\sqrt{2k_t};a)\mathrm{Pe}(v\sqrt{2k_t};-a)$, where $\mathrm{Pe}(\cdot)$ is a parabolic cylinder function of parameter a and even parity.[12] Because of the completeness of these fundamental families, beams with arbitrary polarization can be built up with a suitable superposition of vHzG beams.

It is important to discuss the required conditions for satisfying the paraxial approximation. In the case of the Gaussian-like beams considered by Lax *et al.*,[5] there is an unique characteristic length w_0 in the transverse dimension, and the fields are expanded in powers of the parameter $1/kw_0$. The transverse fields are of order $(1/kw_0)^0$; the longitudinal fields are of order $(1/kw_0)^1$. Thus the paraxial approximation holds when $kw_0 \gg 1$, i.e., when the beam is many wavelengths wide. In the case of the vHzG beams there are two transverse characteristic lengths, namely, w_0 for the Gaussian envelope and $1/k_t$ for function $\boldsymbol{\Psi}$. As a consequence, longitudinal fields $E_z^{(1)}$ and $H_z^{(2)}$ [Eqs. (10b) and (11d)] have not only a term of order $(1/kw_0)^1$ but also a term of order $(k_t/k)^1$. Therefore, to satisfy the paraxial approximation, it is needed that $k \gg 1/w_0$ and additionally that $k \gg k_t$; i.e., the spatial transverse beam oscillations must be many wavelengths wide.

Let us now concentrate our attention on the limiting cases of the vHzG beams. We identify two cases: The first one occurs when $k \gg k_t \gg 1/w_0$, i.e., when the Gaussian width is much larger than the transverse beam oscillations. Under this condition, the second terms in Eqs. (10b) and (11d) can be neglected, and the behavior of $E_z^{(1)}$ and $H_z^{(2)}$ is governed directly by scalar function W. Furthermore, for a very paraxial condition ($kw_0 \ggg 1$), the longitudinal components $H_z^{(1)}$ and $E_z^{(2)}$ are also negligible, and then the first and second-class solutions become purely TM and TE, respectively.

The case when the Gaussian beam size becomes very large ($w_0 \to \infty$) is particularly important. Under this limit $W(X,Y) \to W(x,y)$, $\nabla_T \to \nabla_t$, and the vHzG beams reduce to

$$\mathbf{E}^{\mathrm{TM}} = \exp(ik_z z)\left(\nabla_t W - \hat{\mathbf{z}}\frac{ik_t^2}{k}W\right), \tag{12a}$$

$$\mathbf{H}^{\mathrm{TM}} = \sqrt{\frac{\epsilon_0}{\mu_0}}\exp(ik_z z)(\hat{\mathbf{z}} \times \nabla_t W), \tag{12b}$$

$$\mathbf{E}^{\mathrm{TE}} = -\exp(ik_z z)(\hat{\mathbf{z}} \times \nabla_t W), \tag{12c}$$

$$\mathbf{H}^{\mathrm{TE}} = \sqrt{\frac{\epsilon_0}{\mu_0}}\exp(ik_z z)\left(\nabla_t W - \hat{\mathbf{z}}\frac{ik_t^2}{k}W\right), \tag{12d}$$

which are indeed the equations of the TM and TE vector nondiffracting beams with their longitudinal wave vector k_z expressed in the paraxial approximation $k_z \approx k - k_t^2/2k$. Expressed in circular coordinates, Eqs. (13) describe the mth-order vector Bessel beams introduced by Bouchal and Olivik.[7] Expressed in elliptic and parabolic coordinates, Eqs. (13) correspond to the vector Mathieu and vector parabolic nondiffracting beams, respectively. An interesting property of the vHzG beams in the limit $w_0 \to \infty$ is their correspondence to TM and TE modes in cylindrical waveguides and cylindrical cavities.

The second limiting case of the vHzG beams occurs when $k \gg 1/w_0 \gg k_t$, i.e., when the transverse beam oscillations are much larger than the Gaussian width. Under this condition the first terms in Eqs. (10b) and (11d) are neglected and the behavior of $E_z^{(1)}$ and $H_z^{(2)}$ is then governed by the function $\nabla_T W \cdot \mathbf{r}_t$.

The case when $k_t = 0$ is important. From Eq. (7) we have $Z = 1$; thus the function $\mathbf{U} = \mathbf{\Psi}(X,Y)$ depends only on the transverse coordinates (X,Y). From Eq. (8) it is evident that $\mathbf{\Psi}$ now satisfies the 2D vector Laplace equation $\nabla_T^2 \mathbf{\Psi} = 0$, whose solutions are also given by Eqs. (9), where $W \to \bar{W}(X,Y)$ is now a solution of the scalar Laplace equation $\nabla_T^2 \bar{W} = 0$. Setting $k_t = 0$ in Eqs. (10), the first-class vHzG beams reduce to

$$\mathbf{E}_t^{(1)} = G(\mathbf{r})\nabla_T \bar{W}, \tag{13a}$$

$$E_z^{(1)} = -\frac{2i}{kw_0}\frac{G(\mathbf{r})}{\zeta}\left(\nabla_T \bar{W} \cdot \frac{\mathbf{r}_t}{w_0}\right), \tag{13b}$$

$$\mathbf{H}_t^{(1)} = \sqrt{\frac{\epsilon_0}{\mu_0}}G(\mathbf{r})(\hat{\mathbf{z}} \times \nabla_T \bar{W}), \tag{13c}$$

$$H_z^{(1)} = -\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{2i}{kw_0}\frac{G(\mathbf{r})}{\zeta} \times (\hat{\mathbf{z}} \times \nabla_T \bar{W}) \cdot \frac{\mathbf{r}_t}{w_0}. \tag{13d}$$

Because solutions given by Eqs. (13) are constructed starting from the scalar solutions of the 2D Laplace equation, we have called them (vLpG) beams of the first class. Second-class vLpG beams can be determined from Eqs. (13) by applying the duality property of the solutions of Maxwell equations. Like the Helmholtz equation, the 2D Laplace equation $\nabla_T^2 \bar{W} = 0$ can be solved in several coordinate systems[10] that lead to different functional forms of the vLpG beams. As an example, let us consider the general solution in Cartesian coordinates: $\bar{W} = f(X+iY) + g(X-iY)$, where f and g are arbitrary functions.[10] For the special case $\bar{W} = 0.5(X+iY) + 0.5(X-iY) = X$ we obtain $\nabla_T \bar{W} = \hat{\mathbf{x}}$. By noting first that, in Eqs. (13b) and (13d), for a very paraxial condition ($kw_0 \ggg 1$) the longitudinal components $\{E_z, H_z\}$ are negligible, we see that transverse fields Eqs. (13a) and (13c) reduce to the linearly polarized Gaussian beam that is widely studied within the framework of the scalar theory.

In conclusion, we have demonstrated the existence of the vHzG and vLpG beams that constitute two general families of localized vector beam solutions of the Maxwell equations in the paraxial limit.

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