Ince–Gaussian series representation of the two-dimensional fractional Fourier transform

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We introduce the Ince–Gaussian series representation of the two-dimensional fractional Fourier transform in elliptical coordinates. A physical interpretation is provided in terms of field propagation in quadratic graded-index media whose eigenmodes in elliptical coordinates are derived for the first time to our knowledge. The kernel of the new series representation is expressed in terms of Ince–Gaussian functions. The equivalence among the Hermite–Gaussian, Laguerre–Gaussian, and Ince–Gaussian series representations is verified by establishing the relation among the three definitions.

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The fractional Fourier transform (FrFT) was proposed as a new mathematical tool by Namias in 1980,[1] and subsequently its potential application in optics was explored in 1993 by Ozaktas and Mendlovic[2,3] and Lohmann.[4] Since then a lot of work has been done on the properties, practical implementations, and applications of the FrFT.[5] In an optical context the two-dimensional (2D) FrFT was originally defined by a series of Hermite–Gaussian (HG) functions, and its physical interpretation was provided in terms of field propagation in quadratic graded-index (GRIN) media.[2,3] More recently, Yu *et al.*[6] introduced an alternative series representation of the 2D FrFT in circular cylindrical coordinates whose kernel is constituted by Laguerre–Gaussian (LG) functions. Both definitions are equivalent to each other, because HG and LG functions are the eigenmodes of the GRIN medium in Cartesian and circular cylindrical coordinates, respectively.

Besides the well-known HG beams and LG beams, in recent papers the existence of Ince–Gaussian (IG) beams, which constitute the third complete family of eigenmodes of stable resonators, was theoretically[7–9] and experimentally[10] demonstrated. These new modes are exact and orthogonal solutions of the paraxial wave equation in elliptical coordinates and may be considered continuous transition modes between HG and LG beams.

In this Letter we introduce the IG series representation of the 2D FrFT. The eigenmodes of the GRIN medium in elliptical coordinates are derived for the first time to our knowledge. The transverse distribution of these eigenmodes is described by the IG functions and (like HG and LG eigenmodes) constitutes a complete set of solutions of the 2D Helmholtz equation in a GRIN medium. The kernel of the new representation is expressed in terms of IG functions. The equivalence among the HG, LG, and IG series representations is verified by establishing the relation among the three definitions.

To derive the IG series representation of the 2D FrFT, first we need to determine the eigenmodes of the GRIN medium in elliptical coordinates $\mathbf{r} = (\xi, \eta, z)$. These coordinates are defined as $x = f \cosh \xi \cos \eta$, $y = f \sinh \xi \sin \eta$, and $z = z$, where the radial $\xi \in [0, \infty)$ and the angular $\eta \in [0, 2\pi)$ elliptical coordinates are dimensionless, and semi-focal parameter f has the dimension of length.

Let us consider a GRIN medium whose refractive index varies radially as $n^2(r) = n_0^2(1 - a^2 r^2)$, where $r = (x^2 + y^2)^{1/2}$ is the radial distance from the optical axis and n_0 and a are the GRIN medium parameters. For an optical field $U(\mathbf{r})$ traveling in the positive z direction, the scalar Helmholtz equation is written as

$$[\nabla^2 + k^2(1 - a^2 r^2)]U(\mathbf{r}) = 0 , \tag{1}$$

where $k = n_0 \omega / c$ is the wave number at the optical axis.

Looking for eigenmodes of the GRIN medium in elliptical coordinates, we consider a function of the form

$$U(\mathbf{r}) = \mathrm{IG}(\xi, \eta)\exp(i\beta z) , \tag{2}$$

$$\mathrm{IG}(\xi, \eta) = E(\xi)N(\eta)\exp(-r^2/w^2) , \tag{3}$$

where $E(\xi)$ and $N(\eta)$ are real functions and eigenmode width w and propagation constant β are parameters to be determined.

The existence of eigenmodes of the GRIN medium is ensured if the functions $E(\xi)$ and $N(\eta)$ and parameters w and β can be found such that Eq. (2) satisfies Eq. (1) in elliptical coordinates. Inserting the trial solution, we obtain

$$\frac{\mathrm{d}^2 E}{\mathrm{d}\xi^2} - \epsilon \sinh 2\xi \frac{\mathrm{d}E}{\mathrm{d}\xi} = (\mu - p\epsilon \cosh 2\xi)E , \tag{4}$$

$$\frac{\mathrm{d}^2 N}{\mathrm{d}\eta^2} + \epsilon \sin 2\eta \frac{\mathrm{d}N}{\mathrm{d}\eta} = -(\mu - p\epsilon \cos 2\eta)N , \tag{5}$$

$$w = (2/ak)^{1/2}, \tag{6}$$

$$\beta_p = k[1 - 2a(p + 1)/k]^{1/2}, \tag{7}$$

where p and μ are separation constants and $\epsilon \equiv 2f^2/w^2$ is referred to as the ellipticity parameter of the IG eigenmode.

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Equation (5) is a special case of the Hill equation known as the Ince equation.[8] Notice that we may derive Eq. (4) from Eq. (5) by writing $i\xi$ for η, and vice versa. Solutions of Eq. (5) are known as even and odd Ince polynomials of order p and degree m, usually denoted as $C_p^m(\eta;\epsilon)$ and $S_p^m(\eta;\epsilon)$, respectively, where $0 \le m \le p$ for even functions, $1 \le m \le p$ for odd functions, and indices (p,m) have the same parity, i.e., $(-1)^{p-m} = 1$.

Collecting the partial solutions provides the expression of the IG eigenmodes in GRIN media. For even eigenmodes we have

$$\mathrm{IG}_{p,m}^{e}(\xi,\eta;\epsilon) = C\,C_p^m(i\xi;\epsilon)C_p^m(\eta;\epsilon)\exp\left(-\frac{r^2}{w^2}\right), \quad (8)$$

$$\mathrm{IG}_{p,m}^{o}(\xi,\eta;\epsilon) = S\,S_p^m(i\xi;\epsilon)S_p^m(\eta;\epsilon)\exp\left(-\frac{r^2}{w^2}\right), \quad (9)$$

where C and S are normalization constants and superscript indices e and o refer to even and odd parity, respectively. Several theoretical and experimental transverse shapes of IG modes were reported in Refs. 7, 8, and 10. IG functions are orthonormal with respect to the indices and the parity $\sigma = \{e,o\}$, i.e., $\iint_{-\infty}^{\infty}\mathrm{IG}_{p,m}^{\sigma}\mathrm{IG}_{p',m'}^{\sigma'}\mathrm{d}S = \delta_{\sigma\sigma'}\delta_{pp'}\delta_{mm'}$, where $\mathrm{d}S$ is the area differential element across transverse plane (x,y).

The 2D FrFT of order α of a function $f(x,y)$ is denoted as $\mathcal{F}^{\alpha}[f(x,y)]$. Consistent with previous definitions,[2,3] we require that our definition satisfy two postulates. First, $\mathcal{F}^{1}[f(x,y)]$ should reduce to the conventional Fourier transform, defined as

$$(\mathcal{F}^1 f)(x',y') = \frac{1}{s^2}\iint_{-\infty}^{\infty} f(x,y) \times \exp[-i2\pi(xx'+yy')/s^2]\mathrm{d}S, \quad (10)$$

where x, y, x', y', and s all have dimensions of length and $s = w\pi^{1/2}$. The notation $(\mathcal{F}^1 f)$ emphasizes that variables (x',y') belong to the function $\mathcal{F}^1 f$ and not to f. The second postulate requires the commutative additive property $\mathcal{F}^u\mathcal{F}^v f = \mathcal{F}^v\mathcal{F}^u f = \mathcal{F}^{u+v} f$.

Using the expansion of the IG eigenmodes in terms of LG eigenmodes [Eq. (22), below] and the linearity of the Fourier transform, we found that the eigenvalue equation for the Fourier-transform operator $\mathcal{F}^1$ in elliptical coordinates is given by

$$\mathcal{F}^1[\mathrm{IG}_{p,m}^{\sigma}(\xi,\eta;\epsilon)] = (-i)^p\mathrm{IG}_{p,m}^{\sigma}(\xi',\eta';\epsilon), \quad (11)$$

where the prime elliptical coordinates are $x' = f\cosh\xi'\cos\eta'$ and $y' = f\sinh\xi'\sin\eta'$. In a similar way we found that FrFT operator $\mathcal{F}^{\alpha}$ satisfies the eigenvalue equation

$$\mathcal{F}^{\alpha}[\mathrm{IG}_{p,m}^{\sigma}(\xi,\eta;\epsilon)] = (-i)^{p\alpha}\mathrm{IG}_{p,m}^{\sigma}(\xi',\eta';\epsilon). \quad (12)$$

Equation (12) implies that $\mathrm{IG}_{p,m}^{\sigma}(\xi,\eta;\epsilon)$ is the eigenfunction of operator $\mathcal{F}^{\alpha}$ with eigenvalue $(-i)^{p\alpha}$.

Because of the orthogonality and the completeness of the IG functions, any square-integrable function $f(\xi,\eta)$ can be expanded as

$$f(\xi,\eta) = \sum_{\sigma}\sum_{p=0}^{\infty}\sum_{m=0}^{p} A_{p,m}^{\sigma}\mathrm{IG}_{p,m}^{\sigma}(\xi,\eta;\epsilon), \quad (13)$$

where $A_{p,m}^{\sigma}(\epsilon) = \iint_{-\infty}^{\infty} f(\xi,\eta)\mathrm{IG}_{p,m}^{\sigma}(\xi,\eta;\epsilon)\mathrm{d}S$. Consistent with the physical considerations discussed above, we define the 2D FrFT of $f(\xi,\eta)$ of order α in terms of the IG series representation as

$$\mathcal{F}^{\alpha} f = \sum_{\sigma}\sum_{p=0}^{\infty}\sum_{m=0}^{p}(-i)^{p\alpha}A_{p,m}^{\sigma}\mathrm{IG}_{p,m}^{\sigma}(\xi',\eta';\epsilon). \quad (14)$$

Definition (14) is equivalent to the HG series representation defined by Ozaktas and Mendlovic[2,3] and to the LG series representation obtained by Yu *et al.*[6] The conventional first-order Fourier transform is a special case of Eq. (14) when $\alpha = 1$. The kernel of the 2D FrFT in elliptical coordinates can be determined by insertion of coefficients $A_{p,m}^{\sigma}(\epsilon)$ into Eq. (14), and we obtain

$$\mathcal{F}^{\alpha}[f(\xi,\eta)] = \iint_{-\infty}^{\infty} f(\xi,\eta)K(\xi,\eta,\xi',\eta';\epsilon)\mathrm{d}S, \quad (15)$$

where $K = \sum_{\sigma}\sum_{p=0}^{\infty}\sum_{m=0}^{p}(-i)^{p\alpha}\mathrm{IG}_{p,m}^{\sigma}(\xi,\eta)\mathrm{IG}_{p,m}^{\sigma}(\xi',\eta')$.

The series representation equation (14) is now applied to optical wave propagation. Let $f_0(\xi,\eta)$ denote the complex amplitude distribution at the plane $z=0$ of an optical field propagating in the positive z direction through the GRIN medium. We are interested in the field distribution $f(\xi,\eta,z)$ at planes $z>0$. According to Eq. (13), the function $f_0(\xi,\eta)$ can be expanded in terms of IG eigenfunctions. Then it becomes an easy matter to write the field $f(\xi,\eta,z)$, since we know how each of the IG components propagates [Eq. (2)]:

$$f(\xi,\eta,z) = \sum_{\sigma}\sum_{p=0}^{\infty}\sum_{m=0}^{p} A_{p,m}^{\sigma}\mathrm{IG}_{p,m}^{\sigma}(\xi,\eta)\exp(i\beta_p\alpha L), \quad (16)$$

where $z = \alpha L$ and $L = \pi/2a$ is the distance of propagation that provides the first-order Fourier transform.

If the index variation in Eq. (7) is small [i.e., $2a(p+1) \ll k$] so that β_p can be approximated as $\beta_p \simeq k-(p+1)a$, then $\exp(i\beta_p\alpha L) \approx \exp[ik\alpha L - i(p+1)\alpha\pi/2]$. Substituting this approximation into Eq. (16) and considering definition (14), we can rewrite the propagated field as

$$f(\xi,\eta,z) = \exp(ik\alpha L - i\pi\alpha/2)(\mathcal{F}^{\alpha}f_0)(\xi,\eta). \quad (17)$$

It is concluded from Eq. (17) that the propagation of optical fields in GRIN media can be described by the IG series representation of the 2D FrFT as well as by HG and LG series representations.

Let us now examine the relation among IG, LG, and HG series representations of the 2D FrFT. The eigenvalue equations for 2D FrFT operator $\mathcal{F}^{\alpha}$ in

Cartesian[2,3] and plane-polar coordinates[6] are given by

$$\mathcal{F}^{\alpha}[\mathrm{HG}_{n_x,n_y}(x,y)] = (-i)^{(n_x+n_y)\alpha}\mathrm{HG}_{n_x,n_y}(x',y'), \quad (18)$$

$$\mathcal{F}^{\alpha}[\mathrm{LG}^{\sigma}_{n,l}(r,\phi)] = (-i)^{(2n+l)\alpha}\mathrm{LG}^{\sigma}_{n,l}(r',\phi'), \quad (19)$$

where the HG and LG eigenfunctions are

$$\mathrm{HG}_{n_x,n_y}(x,y) = (2^{n_x+n_y-1}\pi n_x!\, n_y!)^{-1/2} H_{n_x}(\sqrt{2}x/w) \times H_{n_y}(\sqrt{2}y/w)\exp(-r^2/w^2), \quad (20)$$

$$\mathrm{LG}^{e,o}_{n,l}(r,\phi) = [4n!/(1+\delta_{0,l})\pi(n+l)!]^{1/2}\begin{Bmatrix}\cos l\varphi \\ \sin l\varphi\end{Bmatrix} \times (\sqrt{2}r/w)^l L_n^l(2r^2/w^2)\exp(-r^2/w^2), \quad (21)$$

where $H_n(\cdot)$ and $L_n^l(\cdot)$ are the Hermite and the generalized Laguerre polynomials, respectively.

The transition from an $\mathrm{IG}^{e,o}_{p,m}$ eigenmode to a $\mathrm{LG}^{e,o}_{n,l}$ eigenmode occurs when the elliptical coordinates tend to the circular cylindrical coordinates, i.e., when $f \to 0$. In this limit the indices of both modes are related as follows: $l = m$ and $n = (p-m)/2$. On the other hand, the transition from an $\mathrm{IG}^{e,o}_{p,m}$ eigenmode into a HG_{n_x,n_y} eigenmode occurs when $f \to \infty$. In this case the indices are related as follows: for even IG modes $n_x = m$ and $n_y = p - m$, whereas for odd IG modes $n_x = m - 1$ and $n_y = p - m + 1$. Notice in eigenvalue equations (12), (18), and (19) that, in these transitions, p takes the exact value needed to ensure that the eigenvalue of the IG eigenmode is the same eigenvalue as that of the corresponding HG or LG eigenmode.

The IG $\Leftrightarrow$ LG expansions are written as $\mathrm{LG}^{e,o}_{n,l} = \sum_{m=0}^{p=2n+l} B_m \mathrm{IG}^{e,o}_{p=2n+l,m}$ and $\mathrm{IG}^{e,o}_{p,m} = \sum_{l,n} B_{l,n}\mathrm{LG}^{e,o}_{n,l}$. Coefficients B are given explicitly by

$$\iint_{-\infty}^{\infty} \mathrm{LG}^{\sigma}_{n,l}\mathrm{IG}^{\sigma'}_{p,m}\mathrm{d}S = \delta_{\sigma'\sigma}\delta_{p,2n+l}(-1)^{n+l+(p+m)/2} \times [(1+\delta_{0,l})\Gamma(n+l+1)n!]^{1/2} \times c^{\sigma}_{(l+\delta_{o,\sigma})/2}(\mu_p^m), \quad (22)$$

where $c^{\sigma}_{(l+\delta_{o,\sigma})/2}(\mu_p^m)$ is the $(l+\delta_{o,\sigma})/2$-th Fourier coefficient of the C_p^m or S_p^m Ince polynomial.[8] Notice that to build up a LG (or HG) eigenmode the constituent IG eigenmodes must have the same eigenvalue. Consequently the expansions among the three families must involve a finite number of degenerate eigenmodes whose indices satisfy the condition $p = 2n + l = n_x + n_y$ for a given p. It is appropriate then to split each family of LG, IG, and HG eigenmodes into subsets of degenerate eigenmodes that share the same eigenvalue and parity about the positive x axis. Each subset is composed of $N_p = (p + 2\delta_{\sigma,e})/2$ (if p is even) or $N_p = (p+1)/2$ (if p is odd) degenerate eigenmodes that form a complete subbasis of orthonormal eigenmodes under which any field with eigenvalue $(-i)^{p\alpha}$ can be expanded. Therefore any eigenmode of a given subset (e.g., an $\mathrm{IG}^{\sigma}_{p,m}$) can be constructed as a linear superposition of the N_p eigenmodes of any of the other two subsets (e.g., $\mathrm{LG}^{\sigma}_{n,l}$ or HG_{n_x,n_y}). The considerations discussed above establish the equivalence among the IG, HG, and LG series representations of the 2D FrFT. Finally, we remark that eigenvalue equations (11) and (12) can be demonstrated by use of Eq. (18) or (19) after we apply the expansion of the IG eigenmodes in terms of HG or LG eigenmodes and the linearity of the Fourier transform, respectively.

In conclusion, we have derived, for the first time to our knowledge, a series representation of a 2D FrFT in elliptical coordinates. This representation is useful in studies of physical systems with elliptical symmetry, such as radial GRIN fibers with elliptical boundaries, laser cavities supporting IG modes,[10] and quantum systems related to a 2D harmonic oscillator.

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