Ince–Gaussian beams in a quadratic-index medium

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The propagation of Ince–Gaussian beams in media where the refractive index varies quadratically with the distance from the optical axis is studied. Explicit expressions for the complex beam parameter and the longitudinal phase shift are derived and discussed. Ince–Gaussian eigenmodes with constant width can be obtained by satisfying a relation between the beam width and the quadratic-medium coefficient. The derivation has included the possibility of propagation of Ince–Gaussian beams in complex lenslike media having quadratic transverse variations of the index of refraction and the gain or loss. © 2005 Optical Society of America

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In addition to the well-known Hermite–Gaussian (HG) beams and Laguerre–Gaussian (LG) beams,[1] in recent papers the existence of the Ince–Gaussian (IG) beams, which constitute the third complete family of transverse eigenmodes of stable resonators, was theoretically[2–4] and experimentally[5] demonstrated. These new modes are exact and orthogonal solutions of the paraxial wave equation in elliptic coordinates and may be considered continuous transition modes between HG beams and LG beams.

In this paper we derive an expression for the propagation of IG beams in a quadratic-index medium. As with HG and LG beams,[6,7] the transverse intensity distribution of the IG beams is shape invariant and its width is a periodic function of the propagation distance. The propagation of the IG beams through quadratic-index media is formulated in terms of a complex beam parameter. The explicit expressions of the periodic beam width and the longitudinal phase shift are derived. Knowledge of the phase shift is necessary for the study of the resonating conditions of IG beams in laser resonators that exhibit a transverse quadratic gain or loss profile and for the study of the propagation of IG beams through birefringent elements.

Let us consider an inhomogeneous medium whose refractive index varies radially as $n(r) = n_0(1 - a^2r^2/2)$. For a paraxial electric field traveling in the z direction we write $E(\mathbf{r}, t) = \Psi(\mathbf{r})\exp(ikz - i\omega t)$, where $k = n_0\omega/c$ is the wave number at the optical axis and $\Psi(\mathbf{r})$ is a slowly varying complex envelope that satisfies the paraxial wave equation

$$\left(\nabla_t^2 + 2ik\frac{\partial}{\partial z} - k^2a^2r^2\right)\Psi(\mathbf{r}) = 0, \tag{1}$$

where $\nabla_t^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transverse Laplacian. In deriving Eq. (1) the approximation $n^2(r) \sim n_0^2(1 - a^2r^2)$ was used.

We construct a solution of Eq. (1) having the form

$$\Psi(\mathbf{r}) = U(\mathbf{r})\exp\left[\frac{ikr^2}{2q(z)}\right], \tag{2}$$

where $q(z)$ is the complex beam parameter to be determined. Substitution of Eq. (2) into the paraxial wave equation (1) yields two equations,

$$\frac{\mathrm{d}q}{\mathrm{d}z} - a^2q^2 - 1 = 0, \tag{3}$$

$$\nabla_t^2 U + 2ik\frac{\partial U}{\partial z} + \frac{2ik}{q}(U + \nabla_t U \cdot \mathbf{r}_t) = 0, \tag{4}$$

with $\nabla_t = \hat{x}\partial/\partial x + \hat{y}\partial/\partial y$ being the transverse gradient and $\mathbf{r}_t = x\hat{x} + y\hat{y}$ the transverse radius vector.

Equation (3) is a nonlinear differential equation whose solution is

$$q(z_2) = \frac{Aq(z_1) + B}{Cq(z_1) + D}, \tag{5}$$

where

$$A = \cos[a(z_2 - z_1)], \qquad B = \sin[a(z_2 - z_1)]/a,$$

$$C = -a\sin[a(z_2 - z_1)], \qquad D = \cos[a(z_2 - z_1)].$$

Expression (5) represents the transformation law for the complex beam parameter of an IG beam propagating through a quadratic-index medium from the plane z_1 to the plane z_2. Since the designation of z_1 is arbitrary, let us choose the waist plane $z_1 = 0$ to be that for which the constant-phase surface is a plane. To satisfy this requirement we need that $\exp[ikr^2/2q(z_1)]$ be purely real and consequently $q(z_1 = 0) = -iz_R$, where $z_R = kw_0^2/2$ is the Rayleigh range of a Gaussian beam. The complex beam parameter $q(z)$ then becomes

$$q(z) = \frac{1}{a}\frac{\sin(az) - iaz_R\cos(az)}{\cos(az) + iaz_R\sin(az)}. \tag{6}$$

By writing $1/q = 1/R + i2/kw^2$, we determine the beam width $w(z)$ and the radius of curvature $R(z)$ of the wave fronts, and we obtain

$$R(z) = \frac{\tan(az) + a^2 z_R^2 \cot(az)}{a(1 - a^2 z_R^2)}, \tag{7}$$

$$w(z) = w_0[1 + \beta \sin^2(az)]^{1/2}, \tag{8}$$

where $\beta = (1 - a^2 z_R^2)/a^2 z_R^2$.

In attempting to obtain solutions of Eq. (4) in elliptical coordinates (ξ, η, z) we will consider a function of the form

$$U(\mathbf{r}) = E(\xi)N(\eta)\exp[iZ(z)], \tag{9}$$

where E, N, and Z are real functions.

In a transverse z plane, we define the elliptic coordinates as $x = f(z)\cosh\xi\cos\eta$, $y = f(z)\sinh\xi\sin\eta$, $z = z$, where $\xi \in [0, \infty)$ and $\eta \in [0, 2\pi)$ are the radial and angular elliptic variables, respectively. At a given z plane, curves of constant ξ are confocal ellipses and curves of constant η are confocal hyperbolas. The semifocal separation f diverges in the same way as the width of the beam, i.e., $f(z) = f_0 w(z)/w_0$, where f_0 is the semifocal separation at the waist plane $z = 0$.

The existence of the IG beams is ensured if three real functions $E(\xi)$, $N(\eta)$, and $Z(z)$ can be found such that Eq. (9) satisfies Eq. (4) in elliptical coordinates. Inserting the trial solution we obtain the three ordinary differential equations

$$\frac{\mathrm{d}^2E}{\mathrm{d}\xi^2} - \epsilon \sinh 2\xi \frac{\mathrm{d}E}{\mathrm{d}\xi} - (\mu - p\epsilon \cosh 2\xi)E = 0, \tag{10}$$

$$\frac{\mathrm{d}^2N}{\mathrm{d}\eta^2} + \epsilon \sin 2\eta \frac{\mathrm{d}N}{\mathrm{d}\eta} + (\mu - p\epsilon \cos 2\eta)N = 0, \tag{11}$$

$$\frac{\mathrm{d}Z}{\mathrm{d}z} - i\left[\frac{1}{R(z)} + i\frac{2(p+1)}{kw^2(z)}\right] = 0, \tag{12}$$

where p and μ are separation constants and $\epsilon = 2f_0^2/w_0^2$ will be referred to as the ellipticity parameter of the IG beam.

If we substitute $R(z)$ and $w(z)$ into Eq. (12) the function $Z(z)$ is determined after integration, and we have

$$\exp[iZ(z)] = \frac{w_0}{w(z)}\exp\left\{-i(p+1)\arctan\left[\frac{\tan(az)}{az_R}\right]\right\}. \tag{13}$$

Equation (11) is a special case of the Hill equation known as the Ince equation.[8] Notice that Eq. (10) may be derived from Eq. (11) by writing $i\xi$ for η, and vice versa. Solutions of Eq. (11) are known as the even and odd Ince polynomials of order p and degree m usually denoted as $C_p^m(\eta, \epsilon)$ and $S_p^m(\eta, \epsilon)$ respectively, where $0 \leq m \leq p$ for even functions, $1 \leq m \leq p$ for odd functions, and the indices (p, m) have the same parity, i.e., $(-1)^{p-m} = 1$. The basic theory and properties of Ince polynomials have been discussed elsewhere.[3,8]

Collecting the partial solutions provides the expression of the full spatial evolution of IG beams in quadratic-index media, namely,

$$\mathrm{IG}_{p,m}^{e}(\mathbf{r}, t) = \frac{\mathcal{C}w_0}{w(z)} C_p^m(i\xi, \epsilon) C_p^m(\eta, \epsilon) \times \exp\left[\frac{ikr^2}{2q(z)}\right]\exp\left[ikz - i(p+1) \times \arctan\left(\frac{\tan(az)}{az_R}\right) - i\omega t\right], \tag{14}$$

where $\mathcal{C}$ is a normalization constant and the superindex e refers to even modes. Odd IG beams $\mathrm{IG}_{p,m}^{o}(\mathbf{r}, t)$ are obtained by writing the odd Ince polynomials $\mathrm{S}_p^m(\cdot, \epsilon)$ and the odd normalization constant $\mathcal{S}$ instead of the even ones.

At a given z plane, the transverse amplitude distribution of the IG beams is proportional to $C_p^m(i\xi, \epsilon)C_p^m(\eta, \epsilon)\exp[-r^2/w^2(z)]$. Several transverse shapes of even IG beams with $\epsilon = 3$ are depicted in Fig. 1. The patterns exhibit clearly the elliptical transverse structure of the IG beams. The index m corresponds to the number of hyperbolic nodal lines, and $(p - m)/2$ is the number of elliptic nodal lines. Regardless of the indices, the width of the beam is proportional to $w(z)$, so that as z increases, the transverse intensity pattern is affected by the overall factor $w_0^2/w^2(z)$ but otherwise is shape invariant. This result follows from the fact that

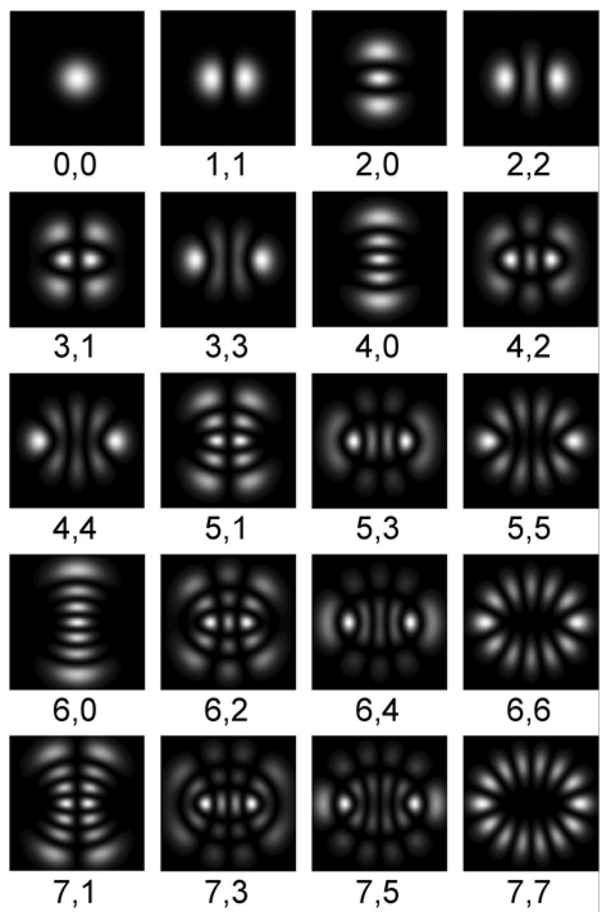

Fig. 1. Transverse field distributions of several even IG beams with $\epsilon = 3$. The IG beam (0, 0) reduces to the simple Gaussian beam. The even IG beam (1, 1) is equal to the HG beam (1, 0) and also to the even LG beam (0, 1).

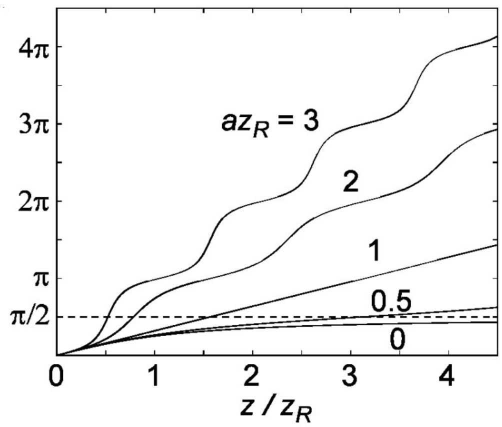

Fig. 2. Longitudinal phase-shift retardation $\phi(z) = \arctan[\tan(az)/az_R]$ as a function of the normalized propagation distance z/z_R for several values of az_R.

the transverse shapes are described in an elliptic coordinate system whose foci evolve according to $f(z) = f_0 w(z)/w_0$ as the beam propagates. The transition from IG beams into LG and HG beams occurs when $f_0 \to 0$ and $f_0 \to \infty$, respectively.[2,3]

Note in Eq. (8) that IG eigenmodes of constant width $w(z) = w_0$ are obtained when $\beta = 0$ or, equivalently, when $w_0 = (2/ak)^{1/2}$. For this propagating scale-invariant eigenmode the wave fronts are exactly plane waves. If the beam width or the wave-front curvature of the input IG beam does not match the IG eigenmode, then the beam will oscillate periodically inward and outward about the mean value

$$\bar{w} = \frac{w_0 a}{2\pi} \int_0^{2\pi/a} [1 + \beta \sin^2(az)]^{1/2} \mathrm{d}z \qquad (15)$$

as it propagates along the axial direction. Equation (14) reduces to the known expression of the IG beams for free-space propagation when a goes to zero.[2,3]

The longitudinal phase shift of the IG beams propagating in a quadratic index medium is given by

$$kz - (p + 1)\phi(z), \qquad (16)$$

where $\phi(z) = \arctan[\tan(az)/az_R]$. The shift then comprises two components: the first, kz, is the phase of a plane wave. The second one is proportional to $\phi(z)$ and represents a phase retardation corresponding to an excess delay of the wave front in comparison with that of a plane wave. The behavior of $\phi(z)$ as a function of the normalized propagation distance z/z_R is depicted in Fig. 2 for several values of az_R. As expected, for free-space propagation ($a = 0$) the total phase retardation as the beam travels from $z = -\infty$ to $z = \infty$ is $(p + 1)\pi$. For quadratic-index media with $a > 0$, the total phase retardation is no longer bounded. For propagating scale-invariant IG eigenmodes ($az_R = 1$), the longitudinal phase shift reduces to a linear function of z, namely, $[k - (p + 1)a]z$.

We remark that IG beams at any z plane are orthonormal with respect to the indices and the parity, i.e.,

$$\int \int_{-\infty}^{\infty} \mathrm{IG}_{p,m}^{\sigma} \overline{\mathrm{IG}}_{p',m'}^{\sigma'} \mathrm{d}S = \delta_{\sigma\sigma'} \delta_{pp'} \delta_{mm'}, \qquad (17)$$

where δ is the Kronecker delta function, the overbar denotes the complex conjugate, and $\sigma = \{e, o\}$ is the parity. Equation (17) is valid when the parameters a and n_0 of the refractive index are real. Any paraxial field traveling in a quadratic-index medium can then be obtained by linear superposition of even and odd IG beams of the form Eq. (14) with the appropriate weighting and phase factors.

From the stationary beam solutions [Eq. (14)] it is possible to construct elliptical rotating IG beams of the form $U^{\pm} = \mathrm{IG}_{p,m}^{e} \pm i\mathrm{IG}_{p,m}^{o}$, where the sign defines the traveling direction.[2,3] This field propagates in the quadratic-index medium carrying orbital angular momentum and exhibiting multiple elliptic vortices, which are attractive properties for potential applications in optical tweezers, particle trapping, and measurements of mechanical torque.

In assuming a complex value for the constants n_0 or a in the expression of the refractive index, we are allowing for the possibility that Eq. (14) describes the propagation of IG beams in quite general quadratic-index media including media exhibiting a nonuniform loss or gain profile—for example, the medium of a gas laser amplifier where the gain decreases with the distance from the axis of the laser tube.[9]

In conclusion, we have derived the complex beam parameter and the longitudinal phase shift of the IG beams propagating in a quadratic-index medium. In the course of obtaining these expressions, we included a derivation of the full spatial evolution of IG beams in complex quadratic-index media. We found that the propagation of the beam can be written in a simple form if the complex beam parameter is used. This parameter gives information on the width of the IG beam and on the curvature of the phase front in each transverse plane of interest. IG eigenmodes with constant width can be obtained by satisfying the input condition $w_0 = (2/ak)^{1/2}$. These IG eigenmodes constitute a complete set of solutions of the two-dimensional Helmholtz equation in a quadratic-index medium and can be used to find the IG series representation of the two-dimensional fractional Fourier transform.[10]

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