Higher-order complex source for elegant Laguerre–Gaussian waves

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We introduce a higher-order complex source that generates elegant Laguerre–Gaussian waves with radial mode number n and angular mode number m. We derive the integral and differential representations for the elegant Laguerre–Gaussian wave that in the appropriate limit yields the corresponding elegant Laguerre-Gaussian beam. From the spectral representation of the elegant Lauguerre–Gaussian wave we determine the first three orders of nonparaxial corrections for the corresponding paraxial elegant Laguerre–Gaussian beam.

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The standard and elegant Hermite–Gaussian,[1,2] Laguerre–Gaussian,[3] and Ince–Gaussian[4,5] beams constitute the three orthogonal and biorthogonal, respectively, complete families of paraxial solutions for the scalar Helmholtz equation. The elegant solutions differ from the standard solutions in that the former contain polynomials with a complex argument, whereas in the latter the argument is real.

There is a considerable and growing interest in extending the model of the paraxial Gaussian beam beyond the paraxial regime. A rigorous method for treating the Gaussian beams in and beyond the paraxial limit is the use of the Felsen method,[6–8] which introduces suitable elementary virtual sources in complex space. The point source in complex space generates in the appropriate region of the physical space the actual paraxial approximation, that is, the Gaussian beam. Shin and Felsen[8] extended this concept to higher-order point sources to generate elegant Hermite–Gaussian beams. Seshadri[9] applied the Felsen method to determine an integral expression for Bessel–Gauss, elegant Hermite–Gaussian, and cylindrical symmetric elegant Laguerre–Gaussian (eLG) waves.

In this Letter we apply the approach introduced by Shin and Felsen to derive a higher-order complex source that generates eLG waves with radial mode number n and angular mode number m. Using this higher-order complex source, we find the integral and differential or multipole representations for eLG waves. From the integral representation we determine the paraxial approximation and the first three orders of nonparaxial corrections for an eLG beam.

Let $U_{n,m}(r)$ be a monochromatic scalar wave function that describes a higher-order eLG wave that propagates along the positive z axis of a cylindrical coordinate system $\mathbf{r} = (r, \phi, z)$. Function $U_{n,m}(\mathbf{r})$ satisfies the homogeneous Helmholtz equation for $z > 0$. The field is assumed to have the general form

$$U_{n,m}(\mathbf{r}) = F_{n,m}(r,z)\exp(\pm im\phi), \tag{1}$$

where the angular phase factor $\exp(\pm im\phi)$ represents the variation in the azimuthal direction. The radial and angular mode numbers of the wave are given by $n = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$, respectively. We assume that $U_{n,m}(\mathbf{r})$ is generated by a higher-order complex source of strength S_{cs} located at $r = 0$ and $z = z_{\mathrm{cs}}$. Parameters S_{cs} and z_{cs} are determined later to yield the desired beam.

Function $F_{n,m}(r,z)$ involves the radial and longitudinal dependence of the wave and satisfies the inhomogeneous Helmholtz equation

$$\left(\nabla_r^2 - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2} + k^2\right)F_{n,m}(r,z) = -S_{\mathrm{cs}}\Theta^m(\nabla_r^2)^n\left[\frac{\delta(r)}{r}\right]\delta(z - z_{\mathrm{cs}}), \tag{2}$$

where k is the wave number, $\delta(\cdot)$ is the Dirac delta function, and the differential operators

$$\nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}, \tag{3}$$

$$\Theta^m = r^m\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^m \tag{4}$$

were derived to ensure that $F_{n,m}(r,z)$ describes an eLG wave with radial n and angular m mode numbers.

A differential equation for the radial spectrum of $F_{n,m}(r,z)$ is determined from Eq. (2) by use of the Hankel transform pair

$$F_{n,m}(r,z) = \int_0^\infty \overline{F}_{n,m}(\alpha,z)J_m(\alpha r)\alpha\,\mathrm{d}\alpha\,, \tag{5a}$$

$$\overline{F}_{n,m}(\alpha,z) = \int_0^\infty F_{n,m}(r,z)J_m(\alpha r)r\,\mathrm{d}r\,, \tag{5b}$$

where $J_m(\cdot)$ is the mth-order Bessel function, $\overline{F}_{n,m}(\alpha,z)$ is the radial spectrum of $F_{n,m}(r,z)$, and α is the radial component of wave vector $\mathbf{k}$.

Substituting the solution for $\overline{F}_{n,m}(\alpha,z)$ into Eq. (5a), we obtain

$$F_{n,m}(r,z) = \int_0^\infty (-1)^n(-\alpha)^{2n+m}\frac{iS_{\mathrm{cs}}}{2\beta} \times \exp[i\beta(z - z_{\mathrm{cs}})]J_m(\alpha r)\alpha\,\mathrm{d}\alpha\,, \tag{6}$$

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for $\mathrm{Re}(z - z_{cs}) > 0$, where $\beta = \beta(\alpha) = (k^2 - \alpha^2)^{1/2}$ is the longitudinal component of wave vector $\mathbf{k}$.

To determine S_{cs} and z_{cs} we now evaluate asymptotically the integral in Eq. (6) under the paraxial regime. For this case α^2 is small compared with k^2, thus we expand β in powers of α^2 and retain the leading term for the amplitude factor and the first two terms for the phase factor. In this approximation Eq. (6) becomes

$$F_{n,m}^{(0)}(r,z) = \frac{iS_{cs}}{2k} \exp[ik(z - z_{cs})] \times \int_0^\infty (-1)^n(-\alpha)^{2n+m} \times \exp\left[-\frac{i\alpha^2}{2k}(z - z_{cs})\right] J_m(\alpha r)\alpha \, \mathrm{d}\alpha \,, \quad (7)$$

where $F_{n,m}^{(0)}$ stands for the zeroth-order (i.e., paraxial) approximation.

To evaluate analytically the integral in Eq. (7) we derived, by applying the operator Θ^m from the left in both sides of Eq. (8) in Ref. 9 and using the differential representation of the associated Laguerre polynomials, the relation

$$\int_0^\infty \alpha^{2n+m} \exp(-p^2\alpha^2) J_m(\alpha r)\alpha \, \mathrm{d}\alpha = \frac{n!}{2} p^{-(2n+m+2)} \left(\frac{r}{2p}\right)^m L_n^m\left(\frac{r^2}{4p^2}\right) \exp\left(-\frac{r^2}{4p^2}\right), \quad (8)$$

where $L_n^m(x)$ is the associated Laguerre polynomial of order n and degree m.

From setting $p^2 = i(z - z_{cs})/2k$, the paraxial field $F_{n,m}^{(0)}(r,z)$ in Eq. (7) becomes

$$F_{n,m}^{(0)}(r,z) = (-1)^{n+m} \frac{iS_{cs}}{2k} \times \exp[ik(z - z_{cs})] \frac{n!}{2} p^{-(2n+m+2)} \times \left(\frac{r}{2p}\right)^m L_n^m\left(\frac{r^2}{4p^2}\right) \exp\left(-\frac{r^2}{4p^2}\right). \quad (9)$$

To obtain the eLG wave for $z > 0$, let us assume that in the paraxial approximation the boundary condition at $z = 0$ is given by the known expression of the eLG beam at the waist plane[3]

$$F_{n,m}^{(0)}(r,0) = (-1)^{n+m} 2^{2n+m} n! \times \left(\frac{r}{w_0}\right)^m L_n^m\left(\frac{r^2}{w_0^2}\right) \exp\left(-\frac{r^2}{w_0^2}\right), \quad (10)$$

where w_0 is the waist size of the beam at the input plane ($z = 0$).

Parameters z_{cs} and S_{cs} are determined by the requirement that Eq. (9) for $z = 0$ reduces to Eq. (10); thus we obtain

$$z_{cs} = ikw_0^2/2 = iz_R \,, \quad (11)$$

$$S_{cs} = -i2z_R w_0^{2n+m} \exp(-kz_R) \,, \quad (12)$$

where $z_R = kw_0^2/2$ is the Rayleigh range of the fundamental Gaussian beam.[1]

Substituting Eqs. (11) and (12) into Eq. (9) yields the paraxial approximation to $F_{n,m}(r,z)$, namely,

$$F_{n,m}^{(0)}(r,z) = (-1)^{n+m} 2^{2n+m} n! \exp(ikz) \times h^{(2n+m+2)} v^{m/2} L_n^m(v) \exp(-v) \,, \quad (13)$$

where

$$h = h(z) = (1 + iz/z_R)^{-1/2}, \quad (14)$$

$$v = v(r,z) = h^2 r^2/w_0^2 \,. \quad (15)$$

Similarly, by substituting z_{cs} and S_{cs} [Eqs. (11) and (12)] into Eq. (6), we obtain the integral expression for $F_{n,m}(r,z)$. After multiplying $F_{n,m}(r,z)$ by the angular phase factor in Eq. (1), we obtain field $U_{n,m}(r)$, namely,

$$U_{n,m}(\mathbf{r}) = (-1)^n \exp(\pm im\phi) z_R \times \exp(-kz_R) w_0^{2n+m} \int_0^\infty (-\alpha)^{2n+m} \beta^{-1} \times \exp[i\beta(z - iz_R)] J_m(\alpha r)\alpha \, \mathrm{d}\alpha \,. \quad (16)$$

Equation (16) is the integral representation for the higher-order eLG. In view of Eq. (11) the source in Eq. (2) lies external to $z > 0$, and the solution given by Eq. (16) is the exact solution to the homogeneous equation corresponding to Eq. (2). The paraxial eLG beam [Eq. (13)] constitutes the lowest-order approximation of the exact eLG wave [Eq. (16)]. However, the integral representation [Eq. (16)] contains all the higher-order nonparaxial contributions and the evanescent waves. For cylindrical symmetric eLG waves, i.e., $m = 0$, Eq. (16) reduces to the special case studied by Seshadri in Ref. 9.

We now apply the Green-function approach to determine the differential representation of the eLG waves. The solution of the differential equation

$$\left(\nabla_r^2 + \frac{\partial^2}{\partial z^2} + k^2\right) G(r,z) = -S_{cs} \frac{\delta(r)}{r} \delta(z - iz_R) \,, \quad (17)$$

is given by[6,7]

$$G(r,z) = S_{cs} \exp(ikR)/2R \,, \quad (18)$$

where $R = [r^2 + (z - iz_R)^2]^{1/2}$. Applying the operator $\Theta^m(\nabla_r^2)^n$ from the left in both sides of Eq. (18) and using the commutation relation

$$\Theta^m \nabla_r^2 - \nabla_r^2 \Theta^m = -\frac{m^2}{r^2} \Theta^m, \quad (19)$$

we get

$$\left(\nabla_r^2 - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\Theta^m(\nabla_r^2)^n G(r,z) = -S_{\mathrm{cs}}\Theta^m(\nabla_r^2)^n\left[\frac{\delta(r)}{r}\right]\delta(z - iz_R). \quad (20)$$

When Eq. (20) is compared with Eqs. (2) and (12), the differential or multipole representation of the eLG waves is found to be

$$U_{n,m}(\mathbf{r}) = -iz_R w_0^{2n+m} \times \exp(-kz_R)\Theta^m(\nabla_r^2)^n \times [\exp(ikR)/R]\exp(\pm im\phi). \quad (21)$$

From Eq. (21) we see that higher-order eLG modes can be generated by applying first the operator $(\nabla_r^2)^n$ and then the operator Θ^m to the fundamental Gaussian solution $\exp(ikR)/R$. While ∇_r^2 increases the radial mode number of the wave field, Θ^m increases its angular mode number.

The scale lengths of variation of r can be seen from Eq. (10) to be w_0. Therefore Eq. (16) indicates that the significant range of variation of α is from 0 to a quantity of the order of $1/w_0$. If the beam waist is large compared with the wavelength, i.e., $1/w_0^2 \ll k^2$, then $\alpha^2 \ll k^2$, which is the requirement used to obtain the paraxial approximation. Thus the existence of the paraxial approximation requires that the beam radius be large compared with the wavelength.

To obtain the first three nonparaxial corrections of the eLG beam, in the exact Eq. (16) we perform the series expansion of $\exp[i\beta(z - iz_R)]$ and β^{-1} in powers of α. In the product of both series only terms up to order $(kw_0)^{-6}$ are retained. The result of this operation enables us to rewrite Eq. (16) as

$$U_{n,m}(\mathbf{r}) \simeq \frac{1}{2}\exp(ikz \pm im\phi)w_0^{(2n+m+2)} \times \int_0^\infty (-1)^n(-\alpha)^{2n+m} \times \exp\left[-\frac{i\alpha^2}{2k}(z - iz_R)\right] \times \sum_{j=0}^{3}\frac{G^{(2j)}(\alpha,z)}{(kw_0)^{2j}} J_m(r\alpha)\alpha\,\mathrm{d}\alpha\,, \quad (22)$$

where

$$G^{(0)}(\alpha, z) = 1\,, \quad (23a)$$

$$G^{(2)}(\alpha, z) = \frac{w_0^2\alpha^2}{2} - \frac{w_0^4\alpha^4}{16h^2}\,, \quad (23b)$$

$$G^{(4)}(\alpha, z) = \frac{3w_0^4\alpha^4}{8} - \frac{w_0^6\alpha^6}{16h^2} + \frac{w_0^8\alpha^8}{512h^4}\,, \quad (23c)$$

$$G^{(6)}(\alpha, z) = \frac{5w_0^6\alpha^6}{16} - \frac{15w_0^8\alpha^8}{256h^2} + \frac{3w_0^{10}\alpha^{10}}{1024h^4} - \frac{w_0^{12}\alpha^{12}}{6\times 4096h^6}. \quad (23d)$$

The integrals in expression (22) can be evaluated by use of Eq. (8) with the result that

$$U_{n,m}(\mathbf{r}) \simeq (-1)^{n+m}2^{2n+m}\exp(ikz \pm im\phi) \times h^{(2n+m+2)}v^{m/2} \times \exp(-v)\sum_{j=0}^{3}\left(\frac{h}{kw_0}\right)^{2j} f_{n,m}^{(2j)}\,, \quad (24)$$

where h and v are given by Eqs. (14) and (15), respectively, and the complex multipliers $f_{n,m}^{(2j)}$ depend on r and z through the following relations:

$$f_{n,m}^{(0)} = n!\,L_n^m(v)\,, \quad (25a)$$

$$f_{n,m}^{(2)} = 2(n+1)!L_{n+1}^m(v) - (n+2)!\,L_{n+2}^m(v)\,, \quad (25b)$$

$$f_{n,m}^{(4)} = 6(n+2)!L_{n+2}^m(v) - 4(n+3)!\,L_{n+3}^m(v) + \frac{1}{2}(n+4)!L_{n+4}^m(v)\,, \quad (25c)$$

$$f_{n,m}^{(6)} = 20(n+3)!L_{n+3}^m(v) - 15(n+4)!\,L_{n+4}^m(v) + 3(n+5)!L_{n+5}^m(v) - \frac{1}{6}(n+6)!L_{n+6}^m(v). \quad (25d)$$

Expression (24) represents the nonparaxial expression for the paraxial eLG beams [Eq. (13)] taking up to the first three nonparaxial corrections.

In conclusion, we have introduced a higher-order complex source that generates eLG waves with azimuthal variation. We derived the integral and differential representation for the eLG waves. From the integral representation we determined in the appropriate limit the paraxial approximation and the first three orders of nonparaxial corrections for the corresponding eLG beam.

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