Elegant Ince–Gaussian beams

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Received March 24, 2004

The existence of elegant Ince–Gaussian beams that constitute a third complete family of exact and biorthogonal elegant solutions of the paraxial wave equation is demonstrated. Their transverse structure is described by Ince polynomials with a complex argument. Elegant Ince–Gaussian beams constitute exact and continuous transition modes between elegant Laguerre–Gaussian and elegant Hermite–Gaussian beams. The expansion formulas among the three elegant families are derived. © 2004 Optical Society of America

OCIS codes: 260.1960, 350.5500, 140.3300, 050.1960, 140.3410.

Hermite–Gaussian beams (HGBs), Laguerre–Gaussian beams (LGBS),[1] and Ince–Gaussian beams[2,3] (IGBs) are eigenmodes of the free-space paraxial wave equation (PWE). The theoretical and practical importance of these standard beams was established mainly because they form three complete bases of orthogonal solutions of the PWE, and additionally they are natural resonating modes in stable laser resonators.[1]

However, an alternative form of the HGBs in which the Hermite polynomials have a complex argument was introduced by Siegman.[4] He called these solutions elegant Hermite–Gaussian beams (eHGBs). The significance of the eHGBs and elegant Laguerre–Gaussian beams (eLGBs) is that they are closely related to multipole fields,[5] they describe propagation through a complex parabolic medium,[1] and they arise in the higher-order connection terms of perturbation expansions of the solutions of the wave equation whose leading term is the fundamental Gaussian beam function.[6,7] The eHGBs and eLGBs each form a biorthogonal set with their adjoint set of functions. Until now, the problem of finding the elegant beams of the PWE in elliptical geometry has remained unexplored.

This Letter introduces elegant Ince–Gaussian beams (eIGBs) that form the third family of exact and biorthogonal elegant solutions of the PWE. The transverse distribution of these fields is described by Ince polynomials[8,9] with a complex argument. Since the eIGBs form a complete biorthogonal set of solutions of the PWE, any paraxial field can be described as a superposition of eIGBs with the appropriate weighting and phase factors. The eHGBs and eLGBs correspond to limiting cases of the eIGBs when the ellipticity parameter tends to infinity or to zero, respectively. The expansion formulas among the three elegant families are also derived.

To derive the eIGBs we proceed as follows: let eIG $(\mathbf{r})$ be the slowly varying complex envelope of a paraxial field that satisfies the PWE $(\nabla_t^2 + 2ik\partial/\partial z)$ eIG $(\mathbf{r}) = 0$, ∇_t^2 is the transverse Laplacian and k is the wave number. Since we expect the usual Gaussian factor $\exp[(ik/2q)r^2]$ to be a basic part of the eigenfunction in any case, we write our proposed eIGB in the form

$$\mathrm{eIG}\,(\mathbf{r}) = A(z)E(\xi)N(\eta)\exp(-cr^2), \tag{1}$$

where A, E, and N are complex functions; r is the radius; $c = c(z) = k/2iq(z)$ is a complex parameter; $q(z) = z - iz_R$; $z_R = kw_0^2/2$ is the Rayleigh range; and w_0 is the beam width at $z = 0$. In a transverse z plane we define the complex elliptical variables as $x = f(z)\cosh\xi\cos\eta$ and $y = f(z)\sinh\xi\sin\eta$, where $f(z) = f_0/w_0c(z)^{1/2}$ and f_0 is the semifocal separation at waist plane $z = 0$, which is the only plane where (ξ, η) variables reduce to the real elliptical coordinates since $w_0c(0)^{1/2} = 1$. It is important to note that (x, y, z) are the real Cartesian coordinates and (ξ, η) are the complex elliptical radial and angular variables, respectively. Putting Eq. (1) into the PWE and making use of $\mathrm{d}q(z)/\mathrm{d}z = 1$ leads to the separated equations

$$\frac{\mathrm{d}^2E}{\mathrm{d}\xi^2} - \epsilon\sinh 2\xi\frac{\mathrm{d}E}{\mathrm{d}\xi} - (a - p\epsilon\cosh 2\xi)E = 0, \tag{2}$$

$$\frac{\mathrm{d}^2N}{\mathrm{d}\eta^2} + \epsilon\sin 2\eta\frac{\mathrm{d}N}{\mathrm{d}\eta} + (a - p\epsilon\cos 2\eta)N = 0, \tag{3}$$

$$\frac{q}{A}\frac{\mathrm{d}A}{\mathrm{d}q} + (p/2 + 1) = 0, \tag{4}$$

where p and a are separation constants and $\epsilon = f_0^2/w_0^2$ is the ellipticity parameter. Solving Eq. (4), we get $A(z) = [q_0/q(z)]^{p/2+1}$, where $q_0 = q(0) = -iz_R$.

Equation (3) is known as the Ince equation.[2,3,8,9] Note that one may derive Eq. (2) from (3) by writing $i\xi$ for η and vice versa. Solutions of Eq. (3) are known as the even and odd Ince polynomials of order p and degree m, usually denoted $C_p^m(\eta, \epsilon)$ and $S_p^m(\eta, \epsilon)$, respectively, where $0 \le m \le p$ for even functions, $1 \le m \le p$ for odd functions, where the indices (p, m) have the same parity, i.e., $(-1)^{p-m} = 1$, where ϵ is the ellipticity parameter.[3,8]

Equation (1) corresponds then to the mathematical description of high-order eIGBs. In a search for three-dimensional solutions, only products of functions of the same parity in ξ and η satisfy continuity in the whole space; thus, rearranging terms provides the general expression of the eIGBs:

$$\mathrm{eIG}^e_{p,m}(\mathbf{r}, \epsilon) = C_e(q_0/q)^{p/2+1}C_p^m(i\xi, \epsilon) \times C_p^m(\eta, \epsilon)\exp(-cr^2), \tag{5}$$

$$\mathrm{eIG}^{o}_{p,m}(\mathbf{r},\epsilon) = S_e(q_0/q)^{p/2+1}S_p^m(i\xi,\epsilon) \times S_p^m(\eta,\epsilon)\exp(-cr^2), \tag{6}$$

where the superscript indices e and o refer to even and odd modes, respectively, and C_e and S_e are normalization constants that will be determined below.

The eIGB are solutions of the equation $\mathcal{L}\mathrm{eIG}^{\sigma}_{p,m} = \lambda_p \mathrm{eIG}^{\sigma}_{p,m}$, where $\sigma = \{e, o\}$ is the parity, $\lambda_p = -2p\epsilon$ is the eigenvalue, and the differential operator $\mathcal{L}$ is

$$\mathcal{L} \equiv \frac{1}{h^2}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right) + \frac{\epsilon}{h^2}\left[\frac{\partial}{\partial\xi}(\sinh 2\xi) - \frac{\partial}{\partial\eta}(\sin 2\eta)\right], \tag{7}$$

where $h^2 = 1/2(\cosh 2\xi - \cos 2\eta)$. This operator is not a Hermitian operator, and its eigenfunctions $\mathrm{eIG}^{\sigma}_{p,m}$ do not form an orthogonal set. The Hermitian adjoint operator $\mathcal{L}^\dagger$ of $\mathcal{L}$ is

$$\mathcal{L}^\dagger \equiv \frac{1}{h^{*2}}\left(\frac{\partial^2}{\partial\xi^{*2}} + \frac{\partial^2}{\partial\eta^{*2}}\right) - \frac{\epsilon}{h^{*2}}\left[(\sinh 2\xi^*)\frac{\partial}{\partial\xi^*} - (\sin 2\eta^*)\frac{\partial}{\partial\eta^*}\right]\cdot \tag{8}$$

The eigenfunctions $\widehat{\mathrm{eIG}}^{\sigma}_{p,m}$ of the adjoint operator are the solutions of the adjoint equation $\mathcal{L}^\dagger\widehat{\mathrm{eIG}}^{\sigma}_{p,m} = \mu_p\widehat{\mathrm{eIG}}^{\sigma}_{p,m}$, where the eigenvalues $\mu_p = \lambda_p$, and are given by

$$\widehat{\mathrm{eIG}}^{e}_{p,m}(\mathbf{r},\epsilon) = C_e(q_0^*/q^*)^{-p/2}C_p^m(i\xi^*,\epsilon)C_p^m(\eta^*,\epsilon), \tag{9}$$

$$\widehat{\mathrm{eIG}}^{o}_{p,m}(\mathbf{r},\epsilon) = S_e(q_0^*/q^*)^{-p/2}S_p^m(i\xi^*,\epsilon)S_p^m(\eta^*,\epsilon). \tag{10}$$

Normalization constants C_e and S_e have been chosen to be equal to the normalization constant of $\mathrm{eIG}^{\sigma}_{p,m}$ to simplify the relations between different elegant bases. Original solutions $\mathrm{eIG}^{\sigma}_{p,m}$ and adjoint solutions $\widehat{\mathrm{eIG}}^{\sigma}_{p,m}$ form a biorthogonal set, with the orthogonality relationship at any z plane given by $(\mathrm{eIG}^{\sigma'}_{p',m'}, \widehat{\mathrm{eIG}}^{\sigma}_{p,m}) = \delta_{\sigma\sigma'}\delta_{pp'}\delta_{mm'}$, where $(F(\mathbf{r}), G(\mathbf{r})) = \iint_{-\infty}^{\infty} F(\mathbf{r})G(\mathbf{r})^* \mathrm{d}x\mathrm{d}y$ and δ is the Kronecker delta function. At the transverse plane $z = 0$, this orthogonality relationship is the same as the one for IGBs with $w_0' = 2^{1/2}w_0$; therefore, $C_e(w_0) = C(2^{1/2}w_0)$ and $S_e(w_0) = S(2^{1/2}w_0)$, where C and S are the normalization constants of the IGBs following the notation of Refs. 2 and 3.

Let us now examine the relationship among eIGBs, eLGBs, and eHGBs. The transition from an $\mathrm{eIG}^{e,o}_{p,m}$ into an $\mathrm{eLG}^{e,o}_{n,l}$ occurs when the elliptical coordinates tend to the circular cylindrical coordinates, i.e, when $f_0 \to 0$. In this limit the indices of both beams are related as follows: $l = m$ and $n = (p - m)/2$. The transition from an $\mathrm{eIG}^{e,o}_{p,m}$ into an eHG_{n_x,n_y} occurs when $f_0 \to \infty$. In this case the indices are related as follows: for even eIGBs, $n_x = m$ and $n_y = p - m$, whereas for odd IGBs, $n_x = m - 1$ and $n_y = p - m + 1$. Figure 1 shows the transition of some eIGBs into their corresponding eLGB and eHGB for both limiting cases. The inherent elliptical geometry of the eIGBs can be appreciated in Fig. 1. As occurs for eHGBs and eLGBs, the amplitude and phase patterns of the higher-order eIGBs change shape with propagation distance z.

Since the three types of elegant beams (eIGBs, eLGBs, eHGBs) form complete biorthogonal sets for

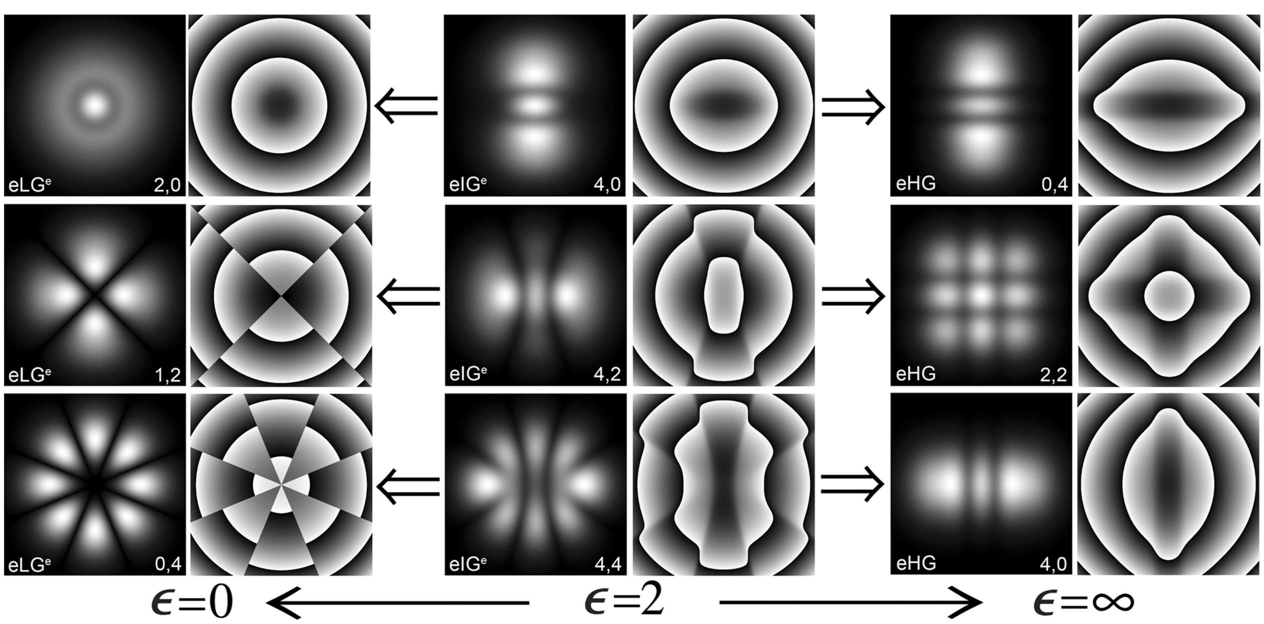

Fig. 1. Transverse amplitudes and phases of eLGBs, eIGBs, and eHGBs at $z = z_R$. The eIGBs correspond to $\epsilon = 2$. eIGBs tend to eLGBs or eHGBs when $\epsilon \to 0$ or $\epsilon \to \infty$, respectively.

expanding an arbitrary paraxial field, one should be able to express one type in terms of any other.

The eHGBs and their adjoint functions are[1]

$$\mathrm{eHG}_{n_x,n_y}(\mathbf{r}) = D_{n_x,n_y}(q_0/q)^{(n_x+n_y)/2+1} H_{n_x}(c^{1/2}x) \times H_{n_y}(c^{1/2}y)\exp(-cr^2), \quad (11)$$

$$\widehat{\mathrm{eHG}}_{n_x,n_y}(\mathbf{r}) = D_{n_x,n_y}(q_0^*/q^*)^{-(n_x+n_y)/2} \times H_{n_x}(c^{*1/2}x)H_{n_y}(c^{*1/2}y), \quad (12)$$

where $H_n(\cdot)$ are the nth-order Hermite polynomials and $D_{n_x,n_y} = (2^{n_x+n_y}\pi n_x!\, n_y!\, w_0^2)^{-1/2}$ is the normalization constant. The biorthogonality relationship at any z plane is given by $(\mathrm{eHG}_{n_x,n_y}, \widehat{\mathrm{eHG}}_{n_x',n_y'}) = \delta_{n_x n_x'}\delta_{n_y n_y'}$.

The eLGBs and their adjoint functions are[6]

$$\mathrm{eLG}_{n,l}^{e,o}(\mathbf{r}) = K_l(q_0/q)^{(2n+l+2)/2}(c^{1/2}r)^l L_n^l(cr^2) \times \exp(-cr^2)\begin{pmatrix}\cos l\phi \\ \sin l\phi\end{pmatrix}, \quad (13)$$

$$\widehat{\mathrm{eLG}}_{n,l}^{e,o}(\mathbf{r}) = K_l(q_0^*/q^*)^{-(2n+l)/2}(c^{*1/2}r)^l \times L_n^l(c^*r^2)\begin{pmatrix}\cos l\phi \\ \sin l\phi\end{pmatrix}, \quad (14)$$

where $L_n^l(\cdot)$ are the generalized Laguerre polynomials and $K_l = [2n!/(1+\delta_{0,l})\pi(n+l)!\,w_0^2]^{1/2}$ is the normalization constant. The biorthogonality relationship at any z plane is given by $(\mathrm{eLG}_{n,l}^{\sigma}, \widehat{\mathrm{eLG}}_{n',l'}^{\sigma'}) = \delta_{\sigma\sigma'}\delta_{nn'}\delta_{ll'}$.

The eIGBs ⇔ eLGBs expansion formulas are given in terms of finite expansions $\mathrm{eIG}_{p,m}^{\sigma} = \sum_{l,n} B_{l,n}\mathrm{LG}_{n,l}^{\sigma}$ and $\mathrm{eLG}_{n,l}^{\sigma} = \sum_{m=0}^{p=2n+1} B_m^* \mathrm{eIG}_{p=2n+l,m}^{\sigma}$. The coefficients B are given explicitly by

$$(\mathrm{eLG}_{n,l}^{\sigma}, \widehat{\mathrm{eIG}}_{p,m}^{\sigma'}) = \delta_{\sigma'\sigma}\delta_{p,2n+l}(i)^{\delta_{o,\sigma}}(-1)^{n+l+(p+m)/2} \times [(1+\delta_{0,l})(n+l)!\,n!]^{1/2} A_{(l+\delta_{o,\sigma})/2}^{\sigma}[a_p^m(\epsilon)], \quad (15)$$

where $A_{(l+\delta_{o,\sigma})/2}^{\sigma}[a_p^m(\epsilon)]$ is the $(l+\delta_{o,\sigma})/2$-th Fourier coefficient of the $C_p^m(\cdot)$ or the $S_p^m(\cdot)$ Ince polynomial.[3,10] With the normalizations proposed in this Letter for eLGBs [Eqs. (13) and (14)] and for eHGBs [Eqs. (11) and (12)] the eLGBs ⇔ eHGBs expansion formulas are the same as those given in Ref. 3 for the LGBs ⇔ HGBs expansion.

It is important to note that beams with the same $p = n_x + n_y = 2n + l$ in any elegant basis have the same z dependence, i.e., $A(z)$ is the same. It seems appropriate, then, to split each family of eIGBs, eLGBs, and eHGBs into subsets of degenerate beams that share the same $p = n_x + n_y = 2n + l$ and the same parity about the positive x axis. Each subset of degenerate beams forms a complete biorthogonal basis under which any elegant Gaussian beam with the same z dependence and parity can be expanded, as can be seen in Eq. (15). As a consequence of this important result, the propagation laws for the eIGBs propagating through any complex paraxial *ABCD* system are the same as those for eHGBs and eLGBs.[1]

A remarkable result of the relationship among the elegant bases is that (given w_0) the four lowest-order, or fundamental, modes each have exactly the same transverse distribution independent of the elegant basis (eIGBs, eHGBs, eLGBs) or the standard basis (IGBs, HGBs, LGBs) used to describe them. This can be explained by noticing that the four lowest-order modes are not degenerate in the z dependence. Table 1 of Ref. 2 includes the indices of these four fundamental modes; the indices for each elegant basis are the same as for its counterpart in the standard basis.

In conclusion, it has been demonstrated that eIGBs are an alternative but equally valid complete family of exact and biorthogonal elegant solutions to the PWE. The eIGBs constitute the exact and continuous transition modes between eLGBs and eHGBs. The propagating characteristics of the three families of elegant Gaussian beams (eIGBs, eLGBs, eHGBs) are restricted to the paraxial regime.

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