Two-dimensional Fourier transform of scaled Dirac delta curves

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We obtain a Fourier transform scaling relation to find analytically, numerically, or experimentally the spectrum of an arbitrary scaled two-dimensional Dirac delta curve from the spectrum of the nonscaled curve. An amplitude factor is derived and given explicitly in terms of the scaling factors and the angle of the forward tangent at each point of the curve about the positive x axis. With the scaling relation we determine the spectrum of an elliptic curve by a circular geometry instead of an elliptical one. The generalization to N-dimensional Dirac delta curves is also included. © 2004 Optical Society of America

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1. INTRODUCTION AND PROBLEM STATEMENT

In optics we often encounter the need for calculating the two-dimensional Fourier transform (FT) of a function $g(x, y)$:

$$\mathcal{F}\{g(x, y)\} = G(u, v) = \int_{-\infty}^{\infty}\int g(x, y)\exp[i(ux + vy)]\mathrm{d}x\mathrm{d}y, \tag{1}$$

where (u, v) are the Cartesian coordinates in the Fourier space and $G(u, v)$ is often referred to as the spectrum of $g(x, y)$. Obtaining an analytical expression for $G(u, v)$ is highly dependent on the function to be transformed; fortunately, enormous amounts of work can be saved by applying the mathematical theorems of the FT.[1]

In particular the scaling theorem states that a stretch of the coordinates in the space domain (x, y) results in a contraction of the coordinates in the Fourier domain (u, v) plus a change in the overall amplitude of the spectrum; that is,

$$\mathcal{F}\{g(\alpha x, \beta y)\} = \frac{1}{|\alpha\beta|}G\left(\frac{u}{\alpha}, \frac{v}{\beta}\right), \tag{2}$$

where α and β are the scaling factors.[1,2] Besides its theoretical importance, the scaling theorem has practical consequences; for example, using it, we may distinguish the FT of an aperture that has the form of an ellipse or a rectangle from that with the form of a circle or a square, respectively.

Of special interest is the case when the function to be transformed is a bright, curvilinear source on the (x, y) plane with an extremely narrow line width. It is common to represent this kind of source of light by the two-dimensional Dirac delta curve, namely, $g(x, y) = \delta(f(x, y))$, where $f(x, y) = 0$ defines an arbitrary curve on the (x, y) plane, and $\delta(\cdot)$ is the Dirac delta function. A classical example of this case occurs in the study of nondiffracting optical beams where the FT of a circular line with constant amplitude leads to the transverse field of a J_0 Bessel beam.[3,4]

Let us assume that the curve $\delta(f(x, y))$ has a known spectrum $G(u, v)$; see Fig. 1(a). In attempting to determine the spectrum of the geometrically scaled curve $\delta(f(\alpha x, \beta y))$, one might be tempted to apply the scaling theorem Eq. (2) and conclude therefore that the spectrum is directly $G(u/\alpha, v/\beta)/|\alpha\beta|$. As we will see later, this result is not correct. When the function to be transformed involves a Dirac delta curve, the scaling theorem as stated by Eq. (2) cannot be applied as straightforwardly as one might assume. Actually, the FT of the scaled Dirac delta curve,

$$\mathcal{F}\{\delta(f(\alpha x, \beta y))\} = W(u, v) = \int_{-\infty}^{\infty}\int \delta(f(\alpha x, \beta y))\exp[i(ux + vy)]\mathrm{d}x\mathrm{d}y, \tag{3}$$

leads to a spectrum $W(u, v)$ that is functionally different from $G(u, v)$, as shown graphically in Fig. 1(b). The need to study this problem arose when we attempted to find the spectrum of an elliptic line by scaling the spectrum of a circular line. By comparing with the rigorous analytical solution, we soon saw clearly that the scaling theorem cannot be directly applied to this case. In testing with some other line contours we obtained similar results.

To illustrate the "failure" of the scaling theorem for the case of Dirac delta curves, let us discuss now a very simple example. Consider a curve $\delta(f(x, y))$ composed of only two straight lines of length $2a$; the first one horizontal and centered at $(0, b)$, the second one vertical and centered at $(b, 0)$, where $a < b$. The function $\delta(f(x, y))$ is written as

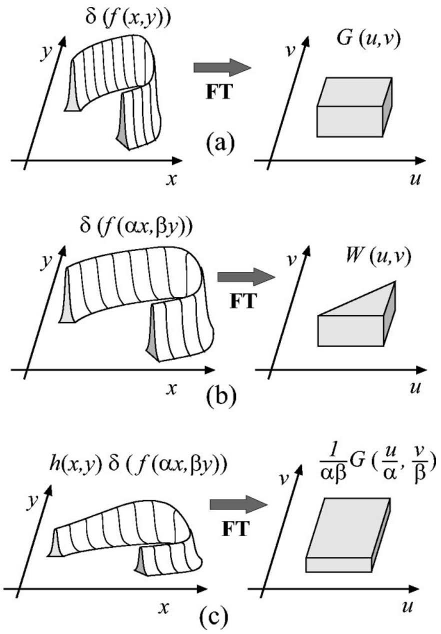

Fig. 1. (a) FT of an arbitrary two-dimensional curve $\delta(f(x, y))$ is given by the spectrum $G(u, v)$. (b) FT of the scaled curve $\delta(f(\alpha x, \beta y))$ is given by the spectrum $W(u, v)$, which is structurally different from $G(u, v)$. (c) FT of the properly modulated curve $h(x, y)\delta(f(\alpha x, \beta y))$ yields the scaled spectrum $G(u/\alpha, v/\beta)/|\alpha\beta|$.

$$\delta(f(x, y)) = \mathrm{rect}\left(\frac{x}{2a}\right)\delta(y - b) + \mathrm{rect}\left(\frac{y}{2a}\right)\delta(x - b), \tag{4}$$

where $\mathrm{rect}(x/2a) = 1$ for $x < |a|$, is equal to 0.5 at $x = a$, and vanishes elsewhere. The spectrum of Eq. (4) can be easily obtained[1]; the result is given by

$$G(u, v) = 2a \exp(ibv)\mathrm{sinc}(au) + 2a \exp(ibu)\mathrm{sinc}(av), \tag{5}$$

where $\mathrm{sinc}\,\theta \equiv \sin\theta/\theta$ is the sinc function. The scaled curve is composed of a horizontal line of length $2a/\alpha$ centered at $(0, b/\beta)$ and a vertical line of length $2a/\beta$ centered at $(b/\alpha, 0)$. The spectrum of $\delta(f(\alpha x, \beta y))$ reads as

$$W(u, v) = \frac{2a}{\alpha}\exp\left(i\frac{bv}{\beta}\right)\mathrm{sinc}\left(\frac{au}{\alpha}\right) + \frac{2a}{\beta}\exp\left(i\frac{bu}{\alpha}\right)\mathrm{sinc}\left(\frac{av}{\beta}\right). \tag{6}$$

From Eqs. (5) and (6) we see clearly that is not possible to determine the spectrum of the scaled curve $\delta(f(\alpha x, \beta y))$ by applying the scaling theorem to the spectrum of the nonscaled curve $\delta(f(x, y))$; in other words $W(u, v) \neq G(u/\alpha, v/\beta)/\alpha\beta$.

By definition, the Dirac delta curve $\delta(f(x, y))$ is mathematically infinite at any point lying on the curve $f(x, y) = 0$ and zero elsewhere. However, for visualization purposes, it is convenient to go back to the definition of the Dirac function as a limit function and consider here that the curve has finite width and height. As we will discuss below, the scaling theorem Eq. (2) is saved by multiplying the scaled curve by a suitable amplitude factor that compensates for the local change of the line "thickness" due to the scaling process. Figure 1(c) shows the scaled Dirac delta curve, but now the height of the Dirac function has been properly modulated.

The purpose of this paper is to present a FT scaling relation for application to two-dimensional Dirac delta curves. The principal result of this work is the derivation of an amplitude factor $h(x, y)$ such that the scaling theorem Eq. (2) may be restated as follows:

$$\mathcal{F}\{h(x, y)\delta(f(\alpha x, \beta y))\} = \frac{1}{|\alpha\beta|}G\left(\frac{u}{\alpha}, \frac{v}{\beta}\right). \tag{7}$$

We have satisfactorily applied the FT scaling relation (7) to a number of situations to find analytically, numerically, and experimentally the spectrum of arbitrary scaled curves $\delta(f(\alpha x, \beta y))$ from the spectra of the nonscaled ones. To our knowledge, this is the first study obtaining the FT of scaled Dirac delta curves. Although our results were conceived for direct application in imaging systems, this is a general formulation and could be used in any area where spectral multidimensional analysis is required.

2. DERIVATION OF THE AMPLITUDE FACTOR

To derive the amplitude factor, we first rewrite Eq. (7) as

$$\mathcal{F}\{\delta(f(\alpha x, \beta y))\} = W(u, v) = \frac{1}{|\alpha\beta|}\mathcal{F}\left\{\mathcal{F}^{-1}\left\{G\left(\frac{u}{\alpha}, \frac{v}{\beta}\right)\right\}H(x, y)\right\}, \tag{8}$$

where $H(x, y) \equiv 1/h(x, y)$. There are two ways in which the factor $H(x, y)$ can be determined: (a) The first one is discussed in this section and is based on the analytical solution of Eq. (8) for a finite, straight Dirac delta segment; (b) the second one is included in Appendix A and is based solely on geometrical considerations.

To find $H(x, y)$ it is adequate to analyze the effect of scaling on just a finite, straight Dirac delta segment. This follows from the fact that the FT is a linear operator; thus any curve may be completely represented as a sequence of differential straight segments.

Let $f_L(x, y) = 0$ define a straight segment of length L making an angle ϕ with the positive x axis, whose middle point is placed at $x = x_0$ and $y = y_0$; see Fig. 2(a). We

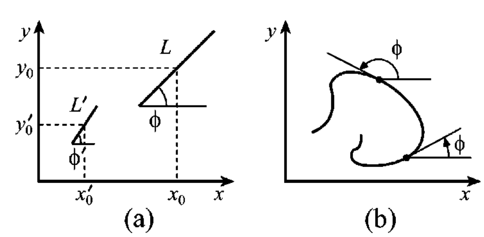

Fig. 2. (a) Effect of scaling a straight segment of length L originally placed at (x_0, y_0) and oriented at an angle ϕ about the positive x axis. (b) At each point, the factor $H(\phi)$ depends on the angle ϕ of the forward tangent measured from the positive x axis.

can find the spectrum $G(u, v)$ of the segment by using the result for a horizontal straight segment at the origin, rotating the axis and then applying the shift theorem of the FT.[1] The result is given by

$$G(u, v) = L \exp(iux_0 + ivy_0) \times \operatorname{sinc}\left[\frac{L}{2}(u \cos\phi + v \sin\phi)\right]. \quad (9)$$

After scaling the spectrum we get

$$\frac{1}{|\alpha\beta|} G\left(\frac{u}{\alpha}, \frac{v}{\beta}\right) = \frac{L}{|\alpha\beta|} \exp\left(\frac{iux_0}{\alpha} + \frac{ivy_0}{\beta}\right) \times \operatorname{sinc}\left[\frac{L}{2}\left(\frac{u\cos\phi}{\alpha} + \frac{v\sin\phi}{\beta}\right)\right]. \quad (10)$$

We now find the FT of the scaled segment. By neglecting the scaling effects on the delta structure we obtain

$$\mathcal{F}\{\delta(f_L(\alpha x, \beta y))\} = W(u, v) = L' \exp(iux_0' + ivy_0') \times \operatorname{sinc}\left(\frac{L'}{2}(u \cos\phi' + v \sin\phi')\right), \quad (11)$$

where $x_0' \equiv x_0/\alpha$, $y_0' \equiv y_0/\beta$, $L' \equiv L[(\cos^2\phi)/\alpha^2 + (\sin^2\phi)/\beta^2]^{1/2}$, $\cos\phi' \equiv (L\cos\phi)/\alpha L'$, and $\sin\phi' \equiv (L\sin\phi)/\beta L'$.

Using Eq. (8) and the fact that for a straight segment the slope is constant and therefore $H(x, y)$ is also constant, we have $W(u, v) = G(u/\alpha, v/\beta)H(x, y)/\alpha\beta$. Finally, using Eqs. (10) and (11) we get $H(x, y) = \alpha\beta L'/L$; thus finally we obtain

$$H(x, y) = H(\phi) = (\beta^2 \cos^2\phi + \alpha^2 \sin^2\phi)^{1/2}. \quad (12)$$

Equation (12) defines the inverse of the amplitude factor $h(x, y)$ that "modulates" each point of the scaled Dirac delta curve $\delta(f(\alpha x, \beta y))$ as required by Eq. (7). Note that at each point of the curve the factor $H(\phi)$ depends solely on the direction angle ϕ of the forward (i.e., counter clockwise-pointing) tangent measured from the positive x axis, as shown in Fig. 2(b).

3. EXAMPLES AND DISCUSSION

Equation (8) provides a way to determine the spectrum of a scaled Dirac delta curve from the spectrum of the nonscaled Dirac delta curve. Despite this expression being theoretically correct, it is not useful from a practical point of view. We can get a more functional expression by performing the inverse FT and writing explicitly the outer FT in the right side of Eq. (8); the result is given by

$$W(u, v) = \frac{1}{|\alpha\beta|} \int_{-\infty}^{\infty}\int H(x, y)\delta(f(x, y)) \times \exp\left[i\left(\frac{u}{\alpha}x + \frac{v}{\beta}y\right)\right] \mathrm{d}x\mathrm{d}y, \quad (13)$$

where we have taken advantage of the fact that the inverse FT of the function G is indeed the nonscaled curve $\delta(f(x, y))$.

There are some important properties of Eq. (13) to be discussed here. First, note that Eq. (13) allows us to calculate the spectrum of the scaled Dirac delta curve $\delta(f(\alpha x, \beta y))$ by integrating over a line corresponding to the nonscaled curve $\delta(f(x, y))$. Second, because of the existence of the amplitude factor, note that Eq. (13) cannot be derived from the original Eq. (3) by applying the simple change of variables $\alpha x \rightarrow x'$ and $\beta y \rightarrow y'$.

To verify the validity of the reformulated scaling theorem Eq. (7), we have applied Eq. (13) for determining analytically, numerically, and experimentally the spectrum of several scaled two-dimensional Dirac delta curves with very good results. For example, it is straightforward to apply Eq. (13) for calculating the spectrum of the two scaled straight lines of Eq. (6) from the spectrum of the nonscaled ones of Eq. (5).

A. Spectrum of an Elliptic Dirac Delta Curve

To apply our formulation on a function that is not explicitly constructed with straight segments, in this example we determine the spectrum of an elliptic delta curve $W_E(u, v)$. Consider an ellipse centered at the origin with semimajor and semiminor axes a and a/β, respectively, with $\beta > 1$ being the scaling factor along the y axis. Finding the spectrum of the ellipse by integrating Eq. (1) directly along an elliptic contour is not a straightforward problem. As shown in Appendix B the solution to this problem requires the use of elliptic coordinates.

Instead of dealing with an elliptic geometry, we take advantage of our formalism to find the spectrum by using a circular contour. For a circular delta with radius a we have $f(x, y) = f(r) = r - a$. Direct substitution of f into Eq. (13) yields

$$W_E(u, v) = \frac{1}{\beta}\int_0^{2\pi}\int_0^{\infty} H(x, y)\delta(r - a) \times \exp\left[i\left(ur\cos\theta + \frac{v}{\beta} r \sin\theta\right)\right] r\mathrm{d}r\mathrm{d}\theta, \quad (14)$$

where the integral has been expressed in cylindrical coordinates (r, θ).

For a circle we have $\tan\phi = -x/y = -\cot\theta$; thus the amplitude factor of Eq. (12) takes the form $H(x, y) = (\beta^2 \sin^2\theta + \cos^2\theta)^{1/2}$. Replacing H in Eq. (14) and performing the integral over r, we obtain

$$W_E(u, v) = \frac{a}{\beta} \int_0^{2\pi} (\beta^2 \sin^2 \theta + \cos^2 \theta)^{1/2} \times \exp\left[i \left(au \cos\theta + \frac{av \sin\theta}{\beta} \right) \right] d\theta. \quad (15)$$

By defining the normalized spatial wavenumbers $\kappa_x \equiv au$, $\kappa_y \equiv av/\beta$ and rearranging, we have

$$W_E(u, v) = a \int_0^{2\pi} (1 - e^2 \cos^2 \theta)^{1/2} \times \exp[iR \cos(\theta - \varphi)] d\theta, \quad (16)$$

where $e^2 = 1 - 1/\beta^2$ is the eccentricity of the ellipse and (R, φ) are the polar coordinates in the (κ_x, κ_y) plane:

$$R \equiv (\kappa_x^2 + \kappa_y^2)^{1/2} = a(u^2 + v^2/\beta^2)^{1/2}, \quad (17a)$$

$$\varphi \equiv \tan^{-1}(\kappa_y/\kappa_x) = \tan^{-1}(v/\beta u). \quad (17b)$$

We have already reached the integral (16) in Appendix B by integrating Eq. (1) directly along an elliptic contour; see Eq. (B5). As demonstrated in Appendix B, the integral (16) is given by

$$W_E(u, v) = \pi a A_0 J_0(R) + 2\pi a \sum_{m=1}^{\infty} (-1)^m A_{2m} (\cos 2m\varphi) J_{2m}(R), \quad (18)$$

where J_m is the mth order Bessel function, $m = 0,1,2,...$, and the coefficients of the series are given in terms of the definite integral

$$A_{2m} = \frac{1}{\pi} \int_0^{2\pi} (\cos 2m\varphi)(1 - e^2 \cos^2 \varphi)^{1/2} d\varphi. \quad (19)$$

Expression (18) gives the exact spectrum of an ellipse. It is known[3,4] that the FT of a circular ring of radius a is $W_C = 2\pi a J_0[a(u^2 + v^2)^{1/2}]$. We see that a naïve application of the scaling theorem to W_C yields only the first term of the series for W_E and consequently ignores the angular dependence of the omitted high-order terms.

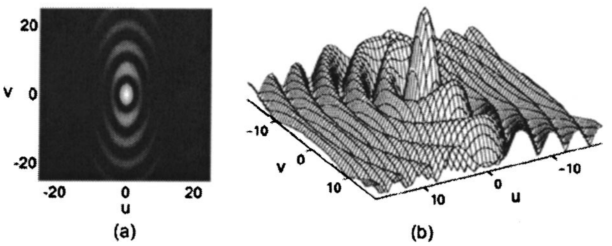

Fig. 3. (a) Absolute value of the transverse shape of the spectrum of an elliptic contour. (b) Amplitude of each ring is not constant but depends on the angular position φ.

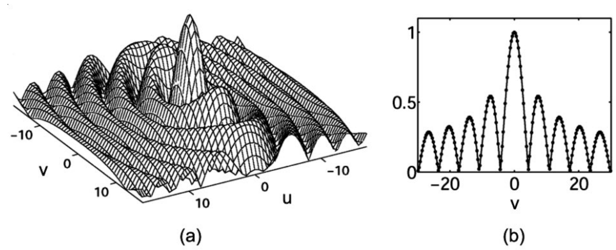

Fig. 4. (a) Absolute value of the transverse shape of the spectrum of an elliptic contour computed with the fast Fourier transform algorithm. (b) Behavior along the v axis of the numerically calculated spectrum (dotted curve) and that computed from the analytical solution of Eq. (18) (solid curve).

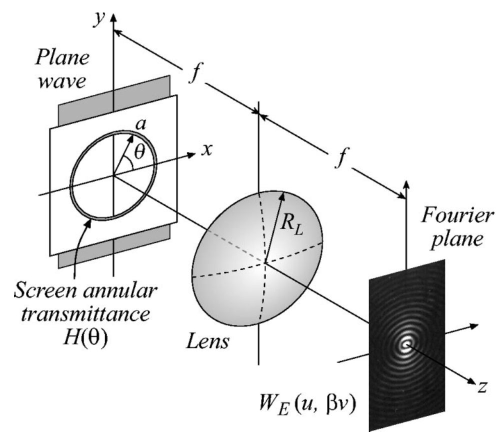

Fig. 5. Experimental setup to measure the scaled power spectrum of an elliptic contour by using a circular contour.

B. Numerical and Experimental Verification

We have applied a variety of methods to determine the spectrum $W_E(u, v)$ of an ellipse with $a = 1$ and $\beta = 2$. Figure 3 corresponds to the absolute value of the analytical expression (18) obtained by adding the first 12 terms of the series. The transverse shape of $W_E(u, v)$ consists of an infinite set of elliptic-like rings with a bright spot at the center.[5] We clearly see in Fig. 3(b) that the amplitude of each ring is not constant but depends on the angular position φ.

The fast Fourier transform algorithm also provides a way to calculate the spectrum W_E by using numerical techniques. We first "draw" an elliptic contour with $\beta = 2$ in a 1024 × 1024 grid matrix[6]; the application of the fast Fourier transform to this matrix yields the pattern shown in Fig. 4(a). The two curves in Fig. 4(b) correspond to the behavior along the v axis of the numerically calculated spectrum (dotted curve) and that computed from the analytical solution of Eq. (18) (solid curve). As we can see, very good agreement between the approaches is obtained.

Experimental verification of the spectrum W_E was realized with the setup shown in Fig. 5. The beam of a 10-mW He–Ne laser at 632.8 nm was spatially filtered by focusing it with a 40× objective into a 5-μm pinhole and then collimated with a lens placed at a focal distance of the pinhole. To generate the annular transmittance, the collimated beam was crossed through photographic film with the required angular variation. This dependence $H(\theta) = (\beta^2 \sin^2 \theta + \cos^2 \theta)^{1/2}$ is manufactured by creating a photographic negative of a gray-scale representation of this amplitude distribution. The photographic film was placed directly in front of the annular circular slit of $2a = 1.55$ mm diameter and 42 μm thickness. At a focal distance f behind the slit a second well-corrected lens of 15 cm focal length was placed.

In physical terms, the arrangement shown in Fig. 5 performs the FT operation according to the definition Eq. (1). We see that Eq. (13) could be interpreted as the FT of the function $H(x, y)\delta(f(x, y))/|\alpha\beta|$ evaluated at the scaled wave numbers u/α and y/β. It is therefore expected that the image in the Fourier plane will be the scaled spectrum $W(\alpha u, \beta v)$. We show in Figs. 6(a) and 6(b) the scaled power spectrum $|W_E(u, \beta v)|^2$ of an ellipse with $\beta = 2$ calculated analytically by Eq. (18).

In Figs. 6(c) and 6(d) we present photographic images of the scaled power spectrum of the ellipse. The pictures were taken at the Fourier plane with a CCD camera and computer frame-grabbing card.[7] A visual comparison with the theoretical predictions in Figs. 6(a) and 6(b) reveals very good agreement. The behavior of the power spectrum $|W_E(u, \beta v)|^2$ along the axes u and v is depicted in Figs. 6(e) and 6(f) for the analytical solution (solid curves) and the experimental measurement (dotted curves).

The transverse pattern in Figs. 6(c) and 6(d) resembles that of nondiffracting fundamental Mathieu beams.[8,9] However, their nature is totally different; Mathieu beams are fundamental solutions of the wave equation in elliptic coordinates, whereas $W_E(u, \beta v)$ is just the spectrum of an ellipse and could be expanded in terms of either Bessel or Mathieu beams.

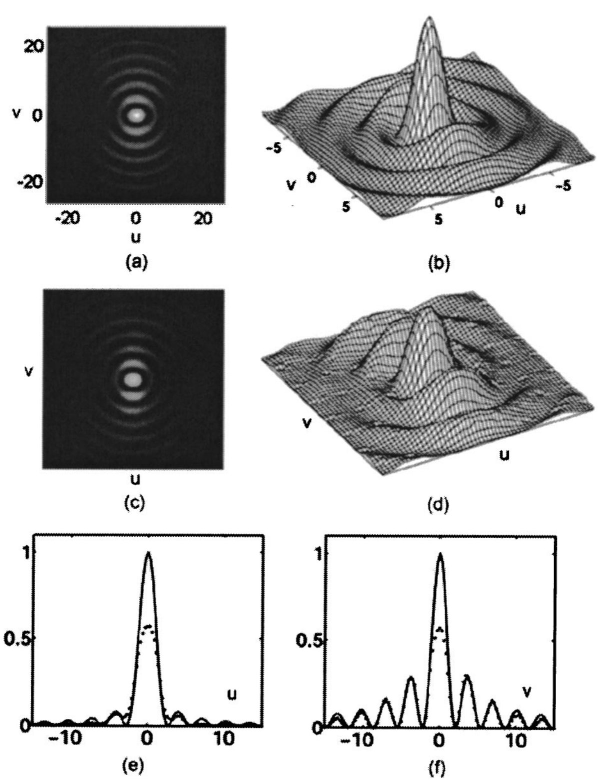

Fig. 6. (a), (b) Theoretical scaled power spectrum of an elliptic contour $|W_E(u, \beta v)|^2$. (c), (d) Image of the transverse intensity pattern of $|W_E(u, \beta v)|^2$ obtained experimentally at the Fourier plane of the setup shown in Fig. 5: (e), (f) Behavior of the power spectrum along the axes u and v for the theoretical solution (solid curves) and the experimental measurement (dotted curves).

4. GENERALIZATION TO *N*-DIMENSIONAL DIRAC DELTA CURVES

The scaling relation Eq. (7) and the Fourier integral Eq. (13) can be straightforwardly generalized to the case of N-dimensional Dirac delta curves. Let $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{u} = (u_1, u_2, \ldots, u_N)$ be the Cartesian coordinates of the N-dimensional position and Fourier spaces, respectively. With this vector notation, the Fourier transform of function $g(\mathbf{x})$ is written as $G(\mathbf{u}) = \int g(\mathbf{x})\exp(i\mathbf{u}\cdot\mathbf{x})\mathrm{d}x^N$, and a Dirac delta curve is represented by $g(\mathbf{x}) = \delta(f(\mathbf{x}))$ in the position space. If the curve is scaled along each dimension by the factors $(\alpha_1, \alpha_2, \ldots, \alpha_N)$, the generalization of Eq. (13) that gives the spectrum of the curve $\delta(f(\alpha_1 x_1, \alpha_2 x_2, \ldots))$ becomes

$$W(\mathbf{u}) = \frac{1}{|\alpha_1\alpha_2\cdots\alpha_N|}\int\cdots\int H(\mathbf{x})\,\delta(f(\mathbf{x})) \times \exp\left[i\left(\frac{u_1 x_1}{\alpha_1} + \frac{u_2 x_2}{\alpha_2} + \cdots\right)\right]\mathrm{d}x^N, \quad (20)$$

where the multiple integral is carried out over the whole space $\mathbf{x}$.

We can find the amplitude factor $H(\mathbf{x})$ for the N-dimensional case by extending either of both derivations of the amplitude factor discussed in Section 2 or Appendix A. The generalized amplitude factor is given by

$$H(\varphi) = |\alpha_1\alpha_2\cdots\alpha_N|\left(\sum_{j=1}^{N}\frac{\cos^2\varphi_j}{\alpha_j^2}\right)^{1/2}, \quad (21)$$

where $\cos\varphi_j = \hat{\mathbf{t}}\cdot\hat{\mathbf{x}}_j$ is the direction cosine of the unit tangent vector $\hat{\mathbf{t}}$ at a given point on the curve $\delta(f(\mathbf{x}))$ with respect to the x_j coordinate axis. From a mathematical point of view, Eq. (20) allows us to calculate the spectrum of the scaled Dirac delta curve $\delta(f(\alpha_1 x_1, \alpha_2 x_2, \cdots))$ by integrating over a line corresponding to the nonscaled curve $\delta(f(\mathbf{x}))$.

5. CONCLUSIONS

A reformulation of the FT scaling theorem for application to two-dimensional Dirac delta curves has been presented. At each point of the original Dirac delta curve, an amplitude factor has been derived by two-different methods and given explicitly in terms of the scale factors and the angle ϕ of the forward tangent measured from the positive x axis. In spite of its simplicity, to our knowledge there is no reference in the literature regarding the use of an alternative scaling theorem for numerically or analytically computing the FT of two-dimensional Dirac delta functions.

We have applied the scaling theorem to determine analytically, numerically, and experimentally the spectrum of an ellipse by using a circular geometry instead of an elliptical one. The three approaches exhibited very good agreement and accuracy. The reformulation presented here is particularly useful when numerically computing a FT; though it may be time-consuming, it eliminates the problems of drawing a two-dimensional Dirac delta function in a rectangular grid, and of the choice of the delta width.

APPENDIX A: GEOMETRIC DERIVATION OF THE AMPLITUDE FACTOR

An alternative derivation of the amplitude factor $H(x, y)$ considering only geometrical scaling effects is included in this appendix for completeness. Consider a straight Dirac delta segment of length L oriented at an angle ϕ with the positive x axis; see Fig. 7(a). For visualization purposes, it is convenient to consider here that the segment has an infinitesimal width τ. The segment can be represented by the limit when $\tau \rightarrow 0$ of a rectangular parallelogram of length L and thickness τ on the (x, y) plane and height $1/\tau$ along the positive z axis. Let the vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ represent the edges of the parallelogram as follows:

$$\mathbf{A} = L[\hat{\mathbf{x}} \cos \phi + \hat{\mathbf{y}} \sin \phi], \tag{A1a}$$

$$\mathbf{B} = \tau[-\hat{\mathbf{x}} \sin \phi + \hat{\mathbf{y}} \cos \phi], \tag{A1b}$$

$$\mathbf{C} = (1/\tau)\hat{\mathbf{z}}. \tag{A1c}$$

The volume within the parallelogram is $\lim_{\tau \rightarrow 0}(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = L$. As shown in Fig. 7(a), after scaling of the axes $x' = \alpha x$ and $y' = \beta y$, the vectors become

$$\mathbf{A}' = L\left(\frac{\cos \phi}{\alpha}\hat{\mathbf{x}} + \frac{\sin \phi}{\beta}\hat{\mathbf{y}}\right), \tag{A2a}$$

$$\mathbf{B}' = \tau\left(-\frac{\sin \phi}{\alpha}\hat{\mathbf{x}} + \frac{\cos \phi}{\beta}\hat{\mathbf{y}}\right). \tag{A2b}$$

The new volume is

$$\lim_{\tau \rightarrow 0}(\mathbf{A}' \times \mathbf{B}') \cdot \mathbf{C} = \frac{L}{\alpha\beta}. \tag{A3}$$

We are interested in keeping the thickness of the delta unchanged, so we define a new width vector $\mathbf{B}''$ with the following properties: $|\mathbf{B}''| = \tau$, $\mathbf{B}''\cdot\mathbf{A}' = 0$, and $\mathbf{B}''\cdot\mathbf{B}' > 0$, where the second property is to ensure that the base of the corrected parallelogram on the (x, y) plane be rectangular; see Fig. 7(b). For this case the volume is given by

$$\lim_{\tau \rightarrow 0}(\mathbf{A}' \times \mathbf{B}'') \cdot \mathbf{C} = \frac{L}{\alpha\beta}(\beta^2 \cos^2 \phi + \alpha^2 \sin^2 \phi)^{1/2}. \tag{A4}$$

We see in Eq. (A4) that the volume of the new parallelogram has an extra factor with respect to Eq. (A3).

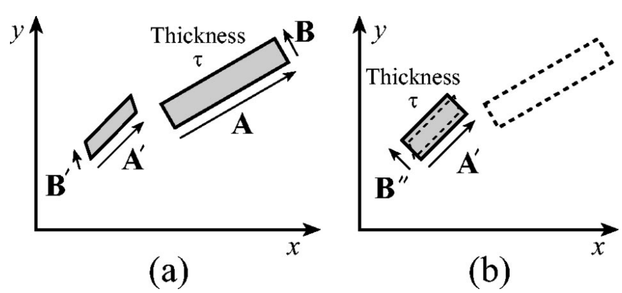

Fig. 7. (a) Scaling changes the rectangular shape of the straight Dirac delta segment. (b) To correct the segment it is necessary to multiply by a suitable amplitude factor.

This is indeed the required factor H that modulates the amplitude of the scaled delta to keep a constant volume. Thus we have

$$H(x, y) = H(\phi) = (\beta^2 \cos^2 \phi + \alpha^2 \sin^2 \phi)^{1/2}, \tag{A5}$$

which, as expected, is Eq. (12). We have seen then that the scaling theorem Eq. (2) may be saved by compensating for the local change of the line "thickness" due to the scaling process with an adjustment of the line "weight" provided by the amplitude factor.

APPENDIX B: ANALYTICAL FOURIER TRANSFORM OF AN ELLIPTIC DIRAC DELTA CURVE

In this appendix we derive the analytic expression of the Fourier transform of an elliptic Dirac delta curve. Consider an ellipse centered at the origin with semimajor axis a and semiminor axis b. We formulate the problem in elliptic coordinates that are defined as

$$x = f \cosh \xi \cos \eta, \tag{B1a}$$

$$y = f \sinh \xi \sin \eta, \tag{B1b}$$

where $\xi \in [0, \infty)$ and $\eta \in [0, 2\pi)$ are the radial and angular elliptic variables and f is the semifocal distance of the ellipse.

Curves of constant ξ are confocal ellipses, and curves of constant η are confocal hyperbolas. The ellipse is defined by $\xi = \xi_0 = \operatorname{arctanh}(b/a) = \text{const.}$, where $a = f \cosh \xi_0$ and $b = f \sinh \xi_0$. The eccentricity e is given by $e = f/a$.

The transformation of the two-dimensional FT (1) to elliptic coordinates yields

$$G(u, v) = \int_0^{2\pi}\int_0^{\infty} g(\xi, \eta)\exp[i(uf \cosh \xi \cos \eta + vf \sinh \xi \sin \eta)]\mathrm{d}x\mathrm{d}y, \tag{B2}$$

where $\mathrm{d}x\mathrm{d}y = f^2(\cosh^2 \xi - \cos^2 \eta)\mathrm{d}\xi\mathrm{d}\eta$ is the differential element of area.

In elliptic coordinates the elliptic Dirac delta is written as

$$g(\xi, \eta) = \frac{\delta(\xi - \xi_0)}{f(\cosh^2 \xi - \cos^2 \eta)^{1/2}}. \tag{B3}$$

Substituting $g(\xi, \eta)$ into Eq. (B2) we obtain

$$G(u, v) = a\int_0^{2\pi}(1 - e^2 \cos^2 \eta)^{1/2} \exp[i(au \cos \eta + bv \sin \eta)]\mathrm{d}\eta. \tag{B4}$$

We now define the normalized spatial wave numbers $\kappa_x \equiv au$ and $\kappa_y \equiv bv$. Rearranging, we have

$$G(u, v) = a\int_0^{2\pi}(1 - e^2 \cos^2 \eta)^{1/2} \exp[iR \cos(\eta - \varphi)]\mathrm{d}\eta, \tag{B5}$$

where $R \equiv (\kappa_x^2 + \kappa_y^2)^{1/2}$ and $\varphi \equiv \tan^{-1}(\kappa_y/\kappa_x)$ can be considered polar coordinates in the normalized Fourier space (κ_x, κ_y).

The integral (B5) can be performed by expanding the radical in Fourier series, namely,

$$(1 - e^2 \cos^2 \eta)^{1/2} = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\eta). \qquad \text{(B6)}$$

The coefficients are given by

$$A_n \equiv \frac{1}{\pi}\int_0^{2\pi} (1 - e^2 \cos^2 \eta)^{1/2} \cos(n\eta)\mathrm{d}\eta, \qquad \text{(B7)}$$

where $A_n = 0$ for $n = 1, 3, 5,...$ because the integrand is even about π.

By taking advantage of the known expansion for the Bessel functions,[10]

$$\int_0^{2\pi} \exp\{i[R\cos(\eta - \varphi)]\}\cos(n\eta)\mathrm{d}\eta = i^n 2\pi[\cos(n\varphi)]\mathrm{J}_n(R), \qquad \text{(B8)}$$

and after rearranging terms, we obtain the exact two-dimensional FT of the elliptic Dirac delta in terms of a Bessel function series, namely

$$G(u, v) = \pi a A_0 \mathrm{J}_0(R) + 2\pi a \sum_{m=1}^{\infty} (-1)^m A_{2m}[\cos(2m\varphi)]\mathrm{J}_{2m}(R), \qquad \text{(B9)}$$

where $m = 1, 2, 3,...,$ $R \equiv (a^2u^2 + b^2v^2)^{1/2}$, $\varphi \equiv \tan^{-1}(bv/au)$, and the coefficients depend on the eccentricity by

$$A_{2m}(e) = \frac{1}{\pi}\int_0^{2\pi} (1 - e^2 \cos^2 \eta)^{1/2} \cos(2m\eta)\mathrm{d}\eta. \qquad \text{(B10)}$$

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5. Strictly speaking, the curves where spectrum W_E vanishes are not ellipses. We would need to solve the equation $W_E(u, v) = 0$ to determine these nodal lines.
6. The proposed scaling theorem of Eq. (7) provides a very efficient way to draw an elliptic Dirac delta curve with constant amplitude in a square grid matrix. By solving for $\delta(f(\alpha x, \beta y))$ we obtain explicitly $\delta(f(\alpha x, \beta y)) = H(x, y)\mathcal{F}^{-1}\{G(u/\alpha, v/\beta)\}/|\alpha\beta|$. It is known that the spectrum G of a circular delta is a zero-order Bessel function; thus the inverse fast Fourier transform of the scaled spectrum $G(u/\alpha, v/\beta)$ yields an elliptic Dirac delta curve whose amplitude is modulated by the factor $h(x, y)$. The effect of multiplying by $H(x, y)$ is to demodulate the Dirac delta curve such that now it has a constant amplitude.
7. It is worth noting that the experimental setup shown in Fig. 5 is indeed the same arrangement commonly used to generate nondiffracting beams (Refs. 3, 4, and 9). We can then take the image of the spectrum not only in the Fourier plane but at any plane within the well-known propagation distance of the nondiffracting beam.
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