Ince–Gaussian beams

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We demonstrate the existence of the Ince–Gaussian beams that constitute the third complete family of exact and orthogonal solutions of the paraxial wave equation. Their transverse structure is described by the Ince polynomials and has an inherent elliptical symmetry. Ince–Gaussian beams constitute the exact and continuous transition modes between Laguerre and Hermite–Gaussian beams. The propagating characteristics are discussed as well. © 2004 Optical Society of America

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Hermite–Gaussian beams (HGB) and Laguerre–Gaussian beams (LGBs) are well-known exact solutions of the free-space paraxial wave equation (PWE) in Cartesian and circular cylindrical coordinates, respectively.[1] Their theoretical and practical importance was established mainly because they form two complete bases of orthogonal modes under which any paraxial optical field can be expanded, and additionally they are natural resonating modes in stable laser resonators.[1] Until now, the problem of finding the exact analytical solutions of the PWE in elliptic cylindrical coordinates had remained unexplored.

We introduce in this Letter the Ince–Gaussian beams (IGBs) that form the third complete family of exact and orthogonal solutions of the PWE, constitute the continuous transition modes between HGBs and LGBs, and are natural resonating modes in stable resonators. The transverse distribution of these fields is described by the Ince polynomials and has an inherent elliptical symmetry. Any paraxial field can be obtained by linear superposition of IGBs with the appropriate weighting and phase factors. LGBs and HGBs correspond to limiting cases of the IGBs when the ellipticity parameter tends to zero or to infinity, respectively.

To derive the IGBs we proceed as follows: Let $\Psi(\mathbf{r})$ be the slowly varying complex envelope of a paraxial field that satisfies the PWE $(\nabla_t^2 + 2ik\partial/\partial z)\Psi(\mathbf{r}) = 0$, where ∇_t^2 is the transverse Laplacian and k is the wave number. The lowest-order Gaussian beam (GB) is $\Psi_G(\mathbf{r}) = [w_0/w(z)]\exp[-r^2/w^2(z) + ikr^2/2R(z) - i\psi_G(z)]$, where r is the radius, w_0 is the beam width at the origin, $w^2(z) = w_0^2(1 + z^2/z_R^2)$ is the beam width, $R(z) = z + z_R^2/z$ is the radius of curvature of the phase front, $\psi_G(z) = \arctan(z/z_R)$ is the Gouy shift, and $z_R = kw_0^2/2$ is the Rayleigh range.[1]

In attempting to obtain solutions of the PWE in elliptical coordinates we consider a wave whose complex envelope is a modulated version of the GB:

$$\mathrm{IG}(\mathbf{r}) = E(\xi)N(\eta)\exp[iZ(z)]\Psi_G(\mathbf{r}), \quad (1)$$

where E, N, and Z are real functions and IG means Ince–Gaussian. In a transverse z plane we define the elliptic coordinates as $x = f(z)\cosh\xi\cos\eta$, $y = f(z)\sinh\xi\sin\eta$, and $z = z$, where $\xi \in [0,\infty)$ and $\eta \in [0, 2\pi)$ are the radial and angular elliptic variables[2] and semifocal separation f diverges in the same way as the width of the GB, i.e., $f(z) = f_0 w(z)/w_0$, where f_0 is the semifocal separation at waist plane $z = 0$.

The existence of the IGBs is ensured if three real functions, $E(\xi)$, $N(\eta)$, and $Z(z)$, can be found such that Eq. (1) satisfies the PWE in elliptical coordinates. Inserting the trial solution into the PWE and using the fact that $\Psi_G(\mathbf{r})$ itself satisfies the PWE, we obtain

$$\frac{\mathrm{d}^2E}{\mathrm{d}\xi^2} - \epsilon\sinh 2\xi\,\frac{\mathrm{d}E}{\mathrm{d}\xi} - (a - p\epsilon\cosh 2\xi)E = 0, \quad (2)$$

$$\frac{\mathrm{d}^2N}{\mathrm{d}\eta^2} + \epsilon\sin 2\eta\,\frac{\mathrm{d}N}{\mathrm{d}\eta} + (a - p\epsilon\cos 2\eta)N = 0, \quad (3)$$

$$-\left(\frac{z^2 + z_R^2}{z_R}\right)\frac{\mathrm{d}Z}{\mathrm{d}z} = p, \quad (4)$$

where p and a are separation constants and $\epsilon = 2f_0^2/w_0^2$ is the ellipticity parameter. From Eq. (4), the excess phase is given by $Z(z) = -p\arctan(z/z_R)$.

Equation (3) is a special case of the Hill equation known as the Ince equation.[3,4] Notice that one may derive Eq. (2) from Eq. (3) by writing $i\xi$ for η and vice versa. This reciprocal relation is important because one may obtain radial solutions $E(\xi)$ from angular solutions $N(\eta)$ by making the argument imaginary. Because angular functions $N(\eta)$ must be 2π periodic, given p and ϵ, Eq. (3) represents an eigenvalue problem whose eigenvalues generate a finite set of real values $a_p^m(\epsilon)$ such that $a_p^0 < \ldots < a_p^p$. For each eigenvalue, the Ince equation admits of so-called finite solutions, i.e., solutions expressible as finite Fourier series, namely, $N(\eta) = \sum A_j(a_p^m)\cos(2j\eta)$.

Solutions of Eq. (3) are known as the even and odd Ince polynomials of order p and degree m, denoted usually $C_p^m(\eta,\epsilon)$ and $S_p^m(\eta,\epsilon)$, respectively, where $0 \le m \le p$ for even functions and $1 \le m \le p$ for odd functions, where the indices (p, m) have the same parity, i.e., $(-1)^{p-m} = 1$, and where ϵ is the ellipticity parameter.[3] Ince polynomials with real argument η are 2π periodic orthogonal functions with m zeros in $0 \le \eta < \pi$ for all values of ϵ.

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Equation (1) corresponds then to the mathematical description of high-order IGBs. In a search for three-dimensional solutions, only products of functions of the same parity in ξ and η satisfy continuity in the whole space; thus, rearranging terms provides the general expression of the IGBs:

$$\mathrm{IG}^{e}_{p,m}(\mathbf{r}) = \frac{Dw_0}{w(z)} C_p^m(i\xi,\epsilon)C_p^m(\eta,\epsilon)\exp\left[\frac{-r^2}{w^2(z)}\right] \times \exp\left[ikz + i\frac{kr^2}{2R(z)} - i(p+1)\arctan\left(\frac{z}{z_R}\right)\right], \tag{5}$$

where D is a normalization constant and the superscript e refers to even modes. We obtain odd IGBs, $\mathrm{IG}^{o}_{p,m}(\mathbf{r})$, by writing S_p^m instead of C_p^m in Eq. (5). IGBs at any z plane are orthonormal with respect to the indices and the parity, i.e., $\iint \mathrm{IG}^{\sigma}_{p,m}\overline{\mathrm{IG}}^{\sigma'}_{p',m'}\mathrm{d}S = \delta_{\sigma\sigma'}\delta_{pp'}\delta_{mm'}$, where δ is the Kronecker delta function and $\sigma = e, o$.

Several transverse field distributions of low-order IGBs at $z = 0$ are shown in Figs. 1 and 2. There are some important physical properties of the IGBs to be discussed here: Notice that m corresponds to the number of hyperbolic nodal lines, whereas $(p - m)/2$ is the number of elliptic nodal lines, without the interfocal nodal line at $\xi = 0$ for the odd modes taken into account. Beams of higher indices have larger widths than those of lower indices. Regardless of the indices, the width of the beam is proportional to $w(z)$, so, as z increases, the transverse intensity pattern is affected by the factor $w_0/w(z)$ but otherwise maintains its profile. Radius of curvature $R(z)$ is the same for all IGBs, implying that IGBs have the same wave fronts and angular divergence as the GB; thus they are focused by a lens and mirrors in precisely the same way. The Gouy phase shift, however, is a function of the order; we obtain $\psi_{\mathrm{IG}}(z) = (p+1)\psi_G(z)$, which means that the phase velocity increases with increasing order number. In resonators this dependence of the phase velocity on the order leads to differences in the resonant frequencies of the various IG modes of oscillation. Like GBs, complex beam parameter $q(z)$ is sufficient for propagating an IGB through a paraxial optical system characterized by an $ABCD$ matrix through the well-known bilinear equation[1] $q_{\mathrm{out}} = (Aq_{\mathrm{in}} + B)/(Cq_{\mathrm{in}} + D)$. Similarly to LGBs and HGBs, the two-dimensional Fourier transform of an IGB has the same transverse structure as the original IGB.

Let us now examine the relationship of IGBs, LGBs, and HGBs. The transition from $\mathrm{IG}^{e,o}_{p,m}$ into $\mathrm{LG}^{e,o}_{n,l}$, where LG means Laguerre–Gaussian, occurs when the elliptic coordinates tend to the circular cylindrical coordinates, i.e., when $f_0 \to 0$. In this limit the indices of both modes are related as follows: $l = m$ and $n = (p - m)/2$. The transition from $\mathrm{IG}^{e,o}_{p,m}$ into HG_{n_x,n_y}, where HG means Hermite–Gaussian, occurs when $f_0 \to \infty$. In this case the indices are related as follows: for even IGBs, $n_x = m$ and $n_y = p - m$, whereas for odd IGBs, $n_x = m - 1$ and $n_y = p - m + 1$. In Fig. 3 we show the transition of an $\mathrm{IG}^{e}_{5,3}$ mode into an $\mathrm{LG}^{e}_{1,3}$ or an $\mathrm{HG}_{3,2}$ mode for both limiting cases. Notice from Eq. (5) that, in the transition, p takes the exact value to ensure that the Gouy shift of the

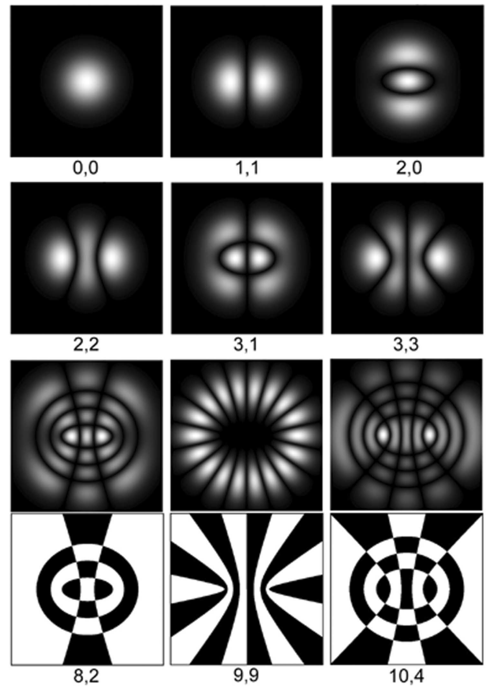

Fig. 1. Transverse field distributions of some even IGBs with $w_0 = 3$ and $\epsilon = 2$. Plots in the bottom row correspond to the phase structures of the modes displayed in the row immediately above them.

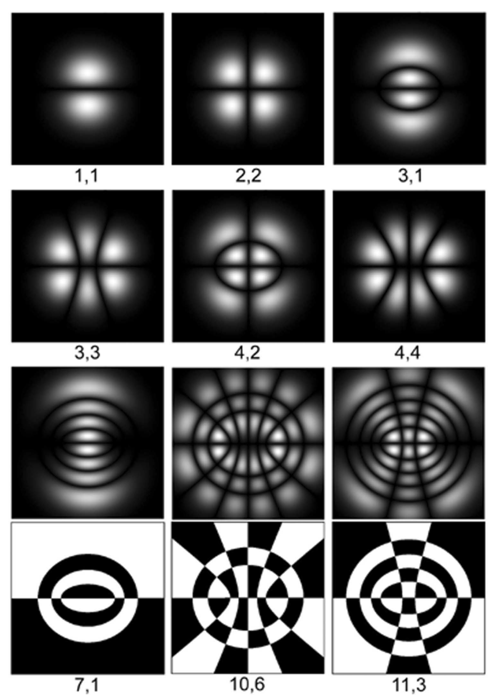

Fig. 2. Same as Fig. 1 but with plots corresponding to odd IGBs.

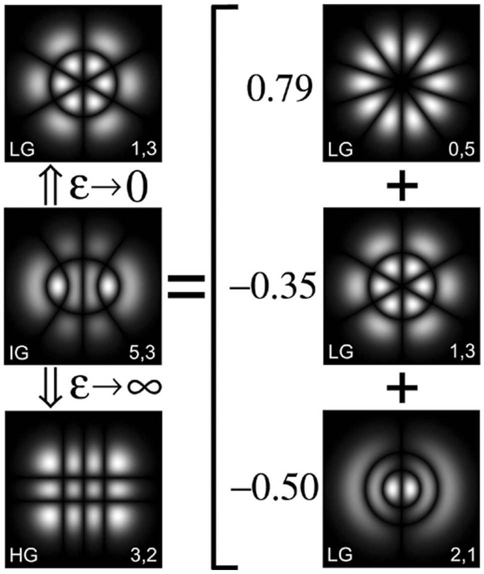

Fig. 3. Left, an IGB tends to a LGB or a HGB when $\epsilon \to 0$ or $\epsilon \to \infty$, respectively. Right, reconstruction of the even IGB, $\mathrm{IG}^e_{5,3} = 0.79\mathrm{LG}^e_{0,5} - 0.35\mathrm{LG}^e_{1,3} - 0.5\mathrm{LG}^e_{2,1}$.

Table 1. The Four Fundamental Modes

Parity	$\mathrm{HG}_{n_x,n_y}(x,y)$	$\mathrm{LG}^\sigma_{n,l}(r,\varphi)$	$\mathrm{IG}^\sigma_{p,m}(\xi,\eta,\epsilon)$
○, $\sigma = e$	0, 0	0, 0	0, 0
∞, $\sigma = e$	1, 0	0, 1	1, 1
8, $\sigma = o$	0, 1	0, 1	1, 1
⊕, $\sigma = o$	1, 1	0, 2	2, 2

IGB becomes the Gouy shift of the corresponding LGB or HGB.[1]

Since three types of mode form complete sets for expanding an arbitrary paraxial field, one should be able to express one type in terms of any other.[5] The IGBs to and from LGB translation formulas are given in terms of the finite expansions $\mathrm{LG}^{e,o}_{n,l} = \sum_{m=0}^{p=2n+l} B_m \mathrm{IG}^{e,o}_{p=2n+l,m}$ and $\mathrm{IG}^{e,o}_{p,m} = \sum_{l,n} B_{l,n}\mathrm{LG}^{e,o}_{n,l}$. The coefficients B are given explicitly by

$$\iint \mathrm{LG}^\sigma_{n,l}\overline{\mathrm{IG}}^{\sigma'}_{p,m}\mathrm{d}S = \delta_{\sigma'\sigma}\delta_{p,2n+l}(-1)^{n+l+(p+m)/2} \times [(1+\delta_{0,l})\Gamma(n+l+1)n!]^{1/2} \times A^\sigma_{(l+\delta_{o,\sigma})/2}(a^m_p)\,, \tag{6}$$

where $A^\sigma_{(l+\delta_{o,\sigma})/2}(a^m_p)$ is the $(l+\delta_{o,\sigma})/2$nd Fourier coefficient of the C^m_p or the S^m_p Ince polynomial.[4,6] Similar expansions can be written for HGBs and IGBs. For instance, for Fig. 3 we obtained mode $\mathrm{IG}^e_{5,3}$ by linearly superposing the three constituent LG modes.

A remarkable result of the relationship among IGBs, LGBs, and HGBs is the existence of four special modes that (given w_0) have exactly the same transverse distribution, independently of the basis used to describe them. This can be explained by the fact that in the translation expansions, for certain combinations of indices, only one constituent mode is needed. In Table 1 we include the pairs of indices of the fundamental modes for each of the three families of Gaussian beams. The e (o) in the first column refers to the even (odd) parity of the mode about the positive x axis. The symbols in the column labeled "Parity" represent the mode shapes.

As can be done with LGBs, from the stationary beam solutions described by Eq. (5) it is possible to construct elliptical rotating fields of the form $U^\pm = \mathrm{IG}^e_{p,m} \pm i\mathrm{IG}^o_{p,m}$, where the sign defines the traveling direction.[7] A field of this kind carries orbital angular momentum and exhibits multiple elliptic vortices,[8] which are attractive properties for potential applications in optical tweezers, particle trapping,[9] and measurements of mechanical torque induced by transitions between LGBs and HGBs.[10,11]

In conclusion, we have demonstrated that an alternative but equally valid family of exact and orthogonal solutions to the PWE can be written in elliptic coordinates rather than in Cartesian or circular cylindrical coordinates. The IGBs constitute the exact and continuous transition modes between LGBs and HGBs and are natural resonating modes in stable laser resonators. The propagating characteristics of the three families of Gaussian beams (HG, LG, and IG) are restricted to the paraxial regime.

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