1 August 2001

OPTICS COMMUNICATIONS

ELSEVIER

Optics Communications 195 (2001) 35–40

www.elsevier.com/locate/optcom

Experimental demonstration of optical Mathieu beams

J.C. Gutiérrez-Vega [a], M.D. Iturbe-Castillo [a], G.A. Ramírez [a], E. Tepichín [a], R.M. Rodríguez-Dagnino [b], S. Chávez-Cerda [c,*,1], G.H.C. New [c]

[a] *Instituto Nacional de Astrofísica, Optica y Electrónica, Apartado Postal 51/216, Puebla, Pue. 72000, Mexico*

[b] *Centro de Electrónica y Telecomunicaciones, Instituto Tecnológico de Estudios Superiores de Monterrey, Monterrey, NL 64849, Mexico*

[c] *Quantum Optics and Laser Science Group, The Blackett Laboratory, Imperial College, London SW7 2BW, UK*

Received 4 May 2001; accepted 10 May 2001

Abstract

We report the first experimental observation of zero-order Mathieu beams which are fundamental non-diffracting solutions of the wave equation in elliptic cylindrical coordinates.

PACS: 42.25.Bs; 42.25.Fx; 42.25.Md

Keywords: Nondiffracting beams; Bessel beams; Mathieu beams; Gaussian beams; Propagation-invariant optical fields; Conical waves; Beam propagation; Electromagnetic waves

Bessel beams (BBs) have been the subject of many studies in recent years, following their experimental observation by Durnin and coworkers [1]. It is now known that these non-diffracting solutions of the wave equation belong to a wider class of propagation-invariant optical fields (PIOFs) which have an extensive range of applications in fields such as metrology, microlithography, medical imaging, non-linear optics and optical and wireless communications among others [2].

A unified description of PIOFs was recently proposed by Salo et al. [3] in the course of an extensive analysis of the problem. PIOFs are commonly represented either as an angular spectrum of plane waves or as a superposition of BB solutions. While the former are the basic solutions of the Helmholtz equation in Cartesian coordinates, the latter are their natural counterparts in circular cylindrical coordinates. Since each forms a complete set, any optical beam can be represented as an appropriate superposition with these functions as the basis.

However, the three-dimensional wave equation is separable in four different *cylindrical* coordinate systems [2] namely, Cartesian (that can be regarded as "rectangular cylindrical"), circular cylindrical, elliptic cylindrical and parabolic cylindrical. In each case, the wave equation separates into a

* Corresponding author. Fax: +44-207-594-7777.
E-mail address: s.chavez@ic.ac.uk (S. Chávez-Cerda).
[1] On leave from INAOE.

[2] In fact, the wave equation is separable in 11 orthogonal coordinate systems, of them four are cylindrical [4].

PII: S0030-4018(01)01319-0

transverse component and a longitudinal component (a basic requirement for PIOFs); this implies that there also exist fundamental transverse modes of the wave equation with elliptical and parabolic structure that belong to the class of PIOFs. These solutions will form complete sets, which can therefore be used as further bases for spectral analysis [5–7].

In this paper, we report the first experimental observation of Mathieu beams that are PIOFs of the Helmholtz equation in elliptic coordinates. The intensity pattern of the zero-order mode is highly localized in one transverse direction, in contrast with the J_0 – BB which is circularly symmetric. However, the physics of Mathieu beams is also based on the superposition of a set of plane waves whose wave vectors lie on a cone [8].

Monochromatic PIOFs are described by the simplified form of Whittaker integral [9–11]

$$u(x,y,z,t) = \exp(\mathrm{i}k_z z - \mathrm{i}\omega t)\int_0^{2\pi} A(\varphi) \times \exp\left[\mathrm{i}k_t(x\cos\varphi + y\sin\varphi)\right]\mathrm{d}\varphi, \tag{1}$$

where φ is the angular variable in the x–y plane, $A(\varphi)$ is an arbitrary function of φ, $k_t = k_0 \sin\theta_0$ and $k_z = k_0\cos\theta_0$ are the respective transverse and longitudinal components of the wave-vector $\mathbf{k}_0$ whose magnitude is ω/c. Eq. (1) represents the superposition of a set of plane waves whose wave vectors lie on a cone of angle $\tan\theta_0 = k_t/k_z$ with respect to the z-axis [12].

We observe that the form of $A(\varphi)$ for *any* monochromatic propagation-invariant field distribution can be computed from Eq. (1) by converting it into a Fredholm integral equation of the first kind, for which several numerical methods exist [13,14].

Since the intersection of the cone of wave vectors with the McCutchen sphere delineates a circumference of radius k_t, this suggests a method to synthesize PIOFs [15]. The experimental setup can be implemented with the help of a thin annular slit modulated by $A(\varphi)$, as was done, for instance, in the experiments described in Refs. [1,14].

It was demonstrated in Ref. [8] that to obtain the modes in elliptic coordinates, the required modulation profiles are the *angular* (periodic) Mathieu functions. For the zero-order mode, we set $A(\varphi) = \mathrm{ce}_0(\varphi; q)$ in Eq. (1) and performed the integral [16]. Omitting a constant factor, the result is the zero-order Mathieu beam

$$u(\xi,\eta,z,t;q) = \mathrm{Je}_0(\xi;q)\mathrm{ce}_0(\eta;q)\exp(\mathrm{i}k_z z - \mathrm{i}\omega t), \tag{2}$$

where Je_0 is the zero-order *radial* Mathieu function of the first kind. The elliptical coordinates (ξ,η,z) are defined by $x = h\cosh\xi\cos\eta$, $y = h\sinh\xi\sin\eta$, $z = z$, where $\xi\in[0,\infty)$ and $\eta\in[0,2\pi)$ are the radial and angular variables, respectively, and $2h$ is the interfocal separation [17]. The parameter $q = h^2k_t^2/4$ carries information about the transverse spatial frequency k_t and the ellipticity of the coordinate system through h. It is clear that Eq. (2) represents a PIOF. Plots of transverse intensity distributions of Mathieu beams are shown in Fig. 1 for $q = 25$ and $q = 595$.

For a circular slit of radius a, the required transmittance $A(\varphi)\delta((x^2+y^2)^{1/2} - a)$ might seem difficult to obtain. However, a mapping of this function to the x-axis reveals that, if we define the variable $X = x/a = \cos\varphi \in [-1,1]$, the normalized mapped transmittance of the upper half of the slit is now $A_X(X;q) = \mathrm{ce}_0(\arccos X;q)/\mathrm{ce}_0(\pi/2;q)$. In Fig. 2 we show the corresponding functions for the beams of Fig. 1 which clearly resemble Gaussians of the form $G(X;q) = \exp[-X^2/2W^2(q)]$, where the width $W(q)$ has to be found.

In order to find $W(q)$, we used a third-order exponential decay method to fit known tabulated values of the function $A_X(X;q)$, and empirically adjusted Gaussian profiles for 75 values of q. We obtained the next fitting function $W(q) = W_0 + W_1\exp[-(q-q_0)/\tau_1] + W_2\exp[-(q-q_0)/\tau_2] + W_3\exp[-(q-q_0)/\tau_3]$ where $W_0 = 0.18065, W_1 = 0.64782, W_2 = 0.16138, W_3 = 0.1822, \tau_1 = 1.78627, \tau_2 = 73.106, \tau_3 = 10.66283$, and $q_0 = -1.09481$. Using the same table, we can also construct $q(W) = q_0 + q_1\exp[-(W-W_0)/\tau_1] + q_2\exp[-(W-W_0)/\tau_2] + q_3\exp[-(W-W_0)/\tau_3]$ in which $q_0 = 1.14672, q_1 = 61.34246, q_2 = 20.15424, q_3 = 77.31582, \tau_1 = 0.09088, \tau_2 = 0.02379, \tau_3 = 0.04364$, and $W_0 =$

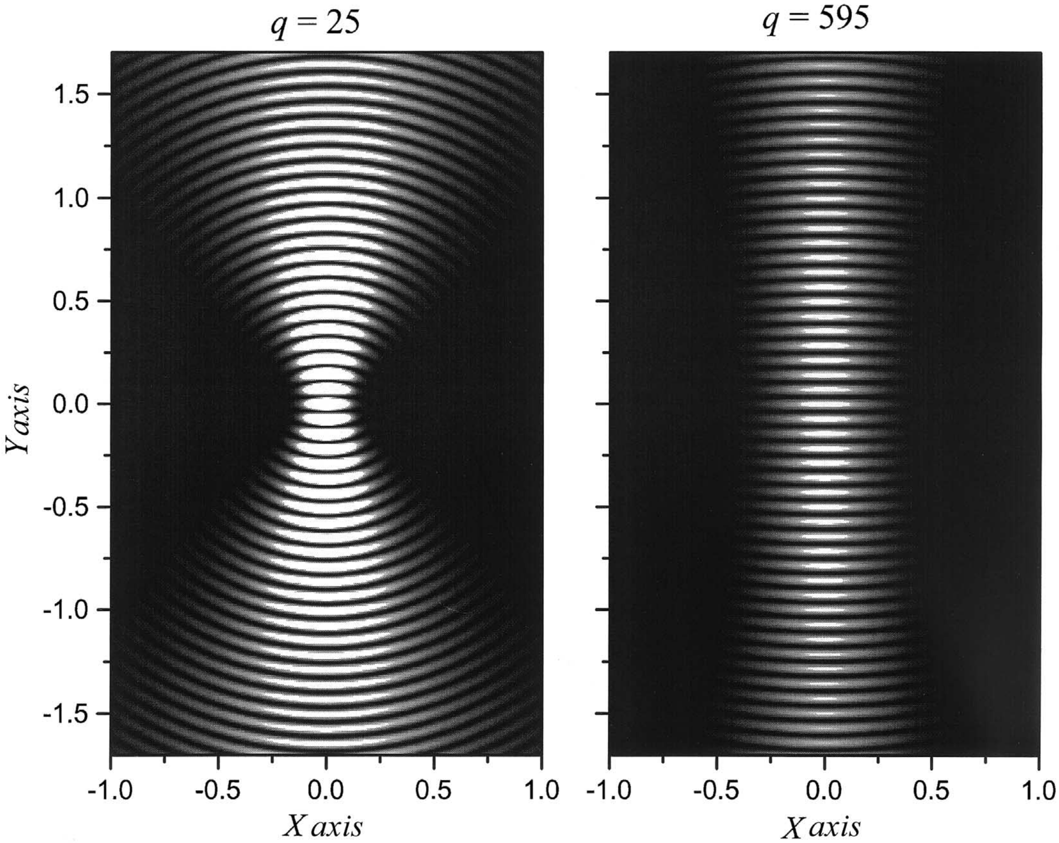

Fig. 1. Theoretical transverse intensity distributions of zero-order Mathieu beams with $q = 25$ and $q = 595$.

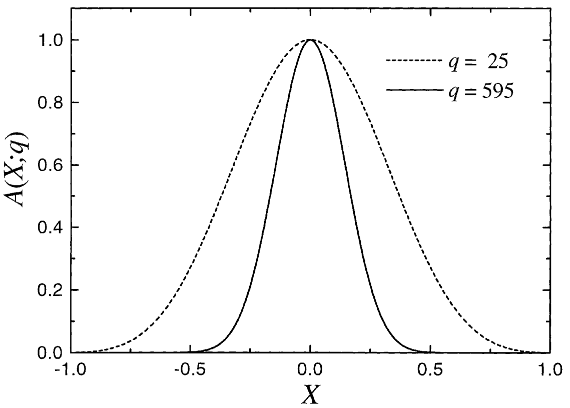

Fig. 2. Corresponding spectra for the two Mathieu beams shown in Fig. 1.

0.1973. Using these relations, the corresponding widths of the Gaussian functions that optimally match the angular spectra shown in Fig. 2 are $W(25) = 0.3098$ and $W(595) = 0.1414$.

Once the required parameters were known, we employed the experimental setup shown in Fig. 3. The beam from a 10 mW He–Ne laser at 6328 Å was spatially filtered by focusing it with a 40× objective into a 5 μm pinhole. By using a well-corrected lens, a collimated beam was formed which passed through a Gaussian aperture of the appropriate width onto an annular slit of 0.1 mm thickness and 3.35 mm radius. Behind the slit there was a second (well-corrected) lens with 75 cm focal length. With this setup, the associated transverse wave number is computed to be $k_t = 44\,349.896\ \mathrm{m}^{-1}$.

In Fig. 4 we present photographic sequences of zero-order Mathieu beams for the two values of the parameter q used above. We kept k_t fixed by

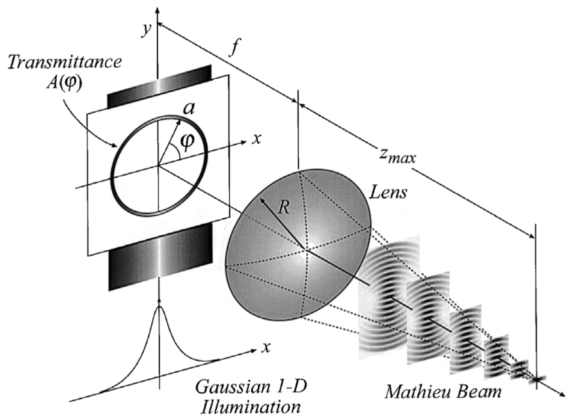

Fig. 3. Experimental setup to generate the zero-order Mathieu beam.

using the same ring-slit and this implies that the width of the pattern will change at its center along the transverse direction defined by the elliptical coordinate system (see explanation below). The pictures were taken at the planes $z = 0$, 1, and 2 m. The first sequence (a)–(c) corresponds to $q = 25$, the second (d)–(f) to $q = 595$. In both cases the patterns remained unchanged over the length of our optical bench ($\sim$4 m). However, the geometrical model predicts a maximum propagation distance of up to 15 m. Notice the excellent agreement with the corresponding computed patterns of Fig. 1.

For a particular physical configuration, i.e. for a fixed k_t, the behavior of the transverse intensity patterns of the Mathieu beams as function of the parameter q are such that the pattern tends to a BB as $q \to 0$ and to a $\cos^2$ pattern for $q \to \infty$. This can be clearly observed in Fig. 4. This situation can be interpreted as follows. From the definition of the parameter q given above, setting $q \to 0$ implies that $h \to 0$, the foci of the elliptical coordinates collapse to a point. In turn, this implies radial symmetry and BB modes. In the other case, the separation of the foci tends to infinity and the pattern will correspond to the interference of plane waves.

Whereas the diffractionless propagation of BBs can be explained by saying that the extensive ring structure maintains the core of the beam, this type of explanation is less convincing for Mathieu beams. In this case, the beam intensity falls rapidly to zero on both sides of the central pillar, a feature that is particularly striking in Fig. 4(d)–(f). Yet, despite this, the interference of the spectral components preserves the width of the pillar and indeed the non-diffracting property of the entire beam.

We also observed in our experiments that Mathieu beams can also reconstruct themselves after they have been partially blocked, as also occurs with BBs. We point out that this feature is common to all the PIOFs defined by Eq. (1) due to their conical wave nature.

The transverse pattern of Mathieu beams resembles that of the bowtie beams, obtained from spatial derivatives of BBs or X-waves solutions with respect to a transverse Cartesian direction [3,20]. However, their physical nature is different since Mathieu beams are fundamental solutions of the wave equation in elliptic cylindrical coordinates; transverse patterns of higher order Mathieu beams can be of two types, namely bowtie-like and elliptical rings [18,19]. Moreover, it can be easily shown that it is also possible to create new solutions in elliptic cylindrical coordinates in the same way as the bowtie beams. This is a consequence of the fact that Eq. (1) is the solution of the wave equation in any cylindrical coordinate system [9–11].

As the paraxial wave equation is also separable in elliptical coordinates, it is expected that its associated solutions share their propagation properties as occurs between Gaussian and BBs to form the Bessel–Gauss beams [21,22]. Such beams have been described as Gaussian modulated conical waves and the propagation of these beams can be understood in good extent by those of paraxial beams traveling along the characteristics of the conical waves [12,23]. These arguments lead to think that Mathieu–Gauss beams solutions of the paraxial wave equation should exist.

In conclusion, we have experimentally demonstrated, for the first time to our knowledge, the possibility of creating approximations to the fundamental mode of the Helmholtz wave equation in elliptical coordinates. This result partially extends the recent unified description of non-diffracting X

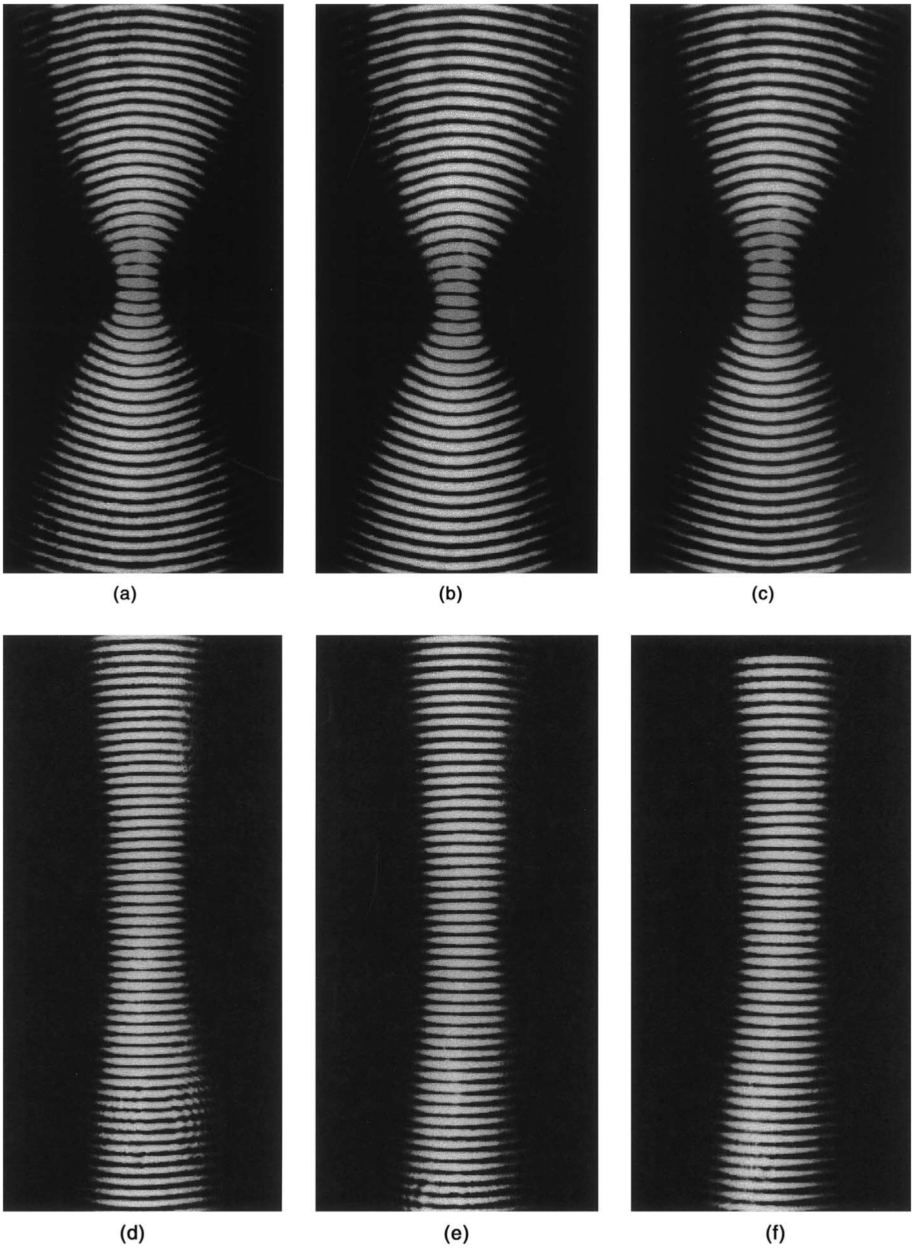

Fig. 4. Photographic sequence of transverse intensity patterns of zero-order Mathieu beams obtained experimentally at $z = 0$, 1 and 2 m for (a–c) $q = 25$, and (d–f) $q = 595$.

and Y waves by including Mathieu beams and their derivatives [3]. The similarity of the zero-order Mathieu beams to bowtie beams and their ease of creation suggest potential applications in medical imaging [20].

Acknowledgements

This work was partially supported by Consejo Nacional de Ciencia y Tecnologia. S. Chávez-Cerda also acknowledges support by the Engineering and Physical Sciences Research Council (grant number GR/N37117/01).

References

[1] J.E. Durnin, J.J. Miceli, J.H. Eberly, Phys. Rev. Lett. 58 (1987) 1499.
[2] J. Salo, J. Fagerholm, A.T. Friberg, M.M. Salomaa, Phys. Rev. Lett. 83 (1999) 1171.
[3] J. Salo, J. Fagerholm, A.T. Friberg, M.M. Salomaa, Phys. Rev. E 62 (2000) 4261.
[4] W. Miller Jr., Symmetry and separation of variables, Encyclopedia of Mathematics and its Applications, vol. 4, Addison-Wesley, Massachusetts, 1977.
[5] G. Gouesbet, L. Mees, G. Gréham, K.F. Ren, Opt. Commun. 161 (1999) 63.
[6] G. Gouesbet, L. Mees, G. Gréham, J. Opt. Soc. Am. A 15 (1998) 3028.
[7] A.A. Inayat-Hussain, J. Math. Phys. 32 (1991) 669.
[8] J.C. Gutiérrez-Vega, M.D. Iturbe-Castillo, S. Chávez-Cerda, Opt. Lett. 25 (2000) 1493.
[9] E.T. Whittaker, Math. Ann. LVII (1902) 333.
[10] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, fourth ed., Cambridge University Press, Cambridge, 1927.
[11] J.A. Stratton, Electromagnetic Theory, McGraw-Hill, New York, 1941.
[12] S. Chávez-Cerda, J. Mod. Opt. 46 (1999) 923–930.
[13] W. Hackbusch, Integral Equations, International Series of Numerical Mathematics, vol. 120, Birkhäuser Verlag, Basel, 1995.
[14] V. Kettunen, J. Turunen, Opt. Lett. 15 (1998) 1247.
[15] G. Indebetouw, J. Opt. Soc. Am. A 6 (1989) 50.
[16] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, fifth ed., Academic Press, London, 1994.
[17] N.W. McLachlan, Theory and Applications of Mathieu Functions, Oxford Press, London, 1951.
[18] J.C. Gutiérrez-Vega, Formal analysis of the propagation of invariant optical fields with elliptical symmetries, Ph.D. Dissertation, Instituto Nacional de Astrofísica, Optica y Electrónica, Puebla, México, 2000; jgutierr@campus.mty.itesm.mx.
[19] J.C. Gutiérrez-Vega, S. Chávez-Cerda, G.H.C. New, Elliptic Vortices of Electromagnetic Wavefields, submitted to Optics Letters.
[20] J. Lu, IEEE Trans. Ultrason. Ferroelectr. Freq. Cont. 42 (1995) 1050.
[21] F. Gori, G. Guattari, C. Padovanni, Opt. Commun. 64 (1987) 491.
[22] R.H. Jordan, D.G. Hall, Opt. Lett. 19 (1994) 427.
[23] S. Chávez-Cerda, G.H.C. New, Opt. Commun. 181 (2000) 369.