Alternative formulation for invariant optical fields: Mathieu beams

J. C. Gutiérrez-Vega, M. D. Iturbe-Castillo, and S. Chávez-Cerda

Instituto Nacional de Astrofisica, Optica y Electrónica, Apartado Postal 51/216, Puebla, Puebla Mexico 72000

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Based on the separability of the Helmholtz equation into elliptical cylindrical coordinates, we present another class of invariant optical fields that may have a highly localized distribution along one of the transverse directions and a sharply peaked quasi-periodic structure along the other. These fields are described by the radial and angular Mathieu functions. We identify the corresponding function in the McCutchen sphere that produces this kind of beam and propose an experimental setup for the realization of an invariant optical field.

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The interest in invariant optical fields (IOF's) is due to the fact that, under ideal conditions, they propagate indefinitely without change of their transverse intensity distribution. Their potential applications in wireless communications, optical interconnections, laser machining, and surgery make them very relevant. However, to create true IOF's would require sources with infinite extent. Nevertheless, in the real world it is possible to generate approximations to such fields, and the distance over which they can propagate without significant alteration is limited by the aperture of the optical system.

A whole class of this kind of wave field was introduced by Durnin *et al.*[1] These fields were described in terms of nonsingular Bessel function J_0. For some applications a ringed structure of Bessel beams can be a disadvantage. For this reason it is important to identify other three-dimensional propagating solutions of the wave equation that have no ringed structure but are still invariant, if they exist.

In group theory it is known that, whenever a partial differential equation is invariant under a continuous symmetry group, one can find a coordinate system in which the equation is separable.[2] In particular, the Helmholtz equation is known to be separable in 11 coordinate systems. Of them, only Cartesian (rectangular cylindrical), circular cylindrical, parabolic cylindrical, and elliptical cylindrical coordinates have translation symmetries such that the equation is separable into a transverse and a longitudinal part. The separability of the equation imposes the condition that the solutions of the transverse part not depend on the longitudinal coordinate. This condition can be trivially observed for plane waves and Bessel beams,[1] and the same must be true for solutions of the Helmholtz equation in any cylindrical coordinate system.

Based on the separability of the Helmholtz equation into elliptical cylindrical coordinates, we present a new analytic formulation of IOF's that may have a highly localized distribution along one of the transverse directions and a sharply peaked quasi-periodic structure along the other. These fields are described by the Mathieu functions that are exact solutions of the Helmholtz equation. We also present a general formalism for creating IOF's in terms of the McCutchen theorem[3,4] that, when it is applied to the solutions found, provides a way to generate Mathieu beams. A simple experimental setup is proposed for the realization of these IOF's.

For optical fields the Helmholtz equation is obtained from the electromagnetic-wave equation, assuming a temporal dependence of the form $\exp(-i\omega t)$. Looking for the fundamental traveling-wave solutions in any of the four cylindrical coordinate systems mentioned above, we find that the simplest of the IOF's is a plane wave whose wave vector has a magnitude $k_0 = \omega/v$, where ω is the wave's frequency and v is its phase velocity.

In a Cartesian frame two interfering plane waves can produce an IOF with a $\cos^2$ transverse intensity pattern that propagates without change in its structure.[5] Further superposition of a finite or an infinite number of plane waves may produce propagation-invariant patterns. The wave vectors of the new set of plane waves must fulfill the conditions imposed by the McCutchen sphere (MS).[3,4] To formalize this condition let us write the three-dimensional amplitude distribution of a scalar optical field in infinite space as[1,6]

$$u(x, y, z) = \exp(ik_z z)\int_0^{2\pi} A(\varphi) \times \exp[ik_t(x\cos\varphi + y\sin\varphi)]\mathrm{d}\varphi\,, \qquad (1)$$

where $A(\varphi)$ is an arbitrary modulation complex function, φ is an angular variable, and $k_t = k_0 \sin\theta_0$ and $k_z = k_0 \cos\theta_0$ are the magnitudes of the transverse and the longitudinal components of the wave vector $\mathbf{k_0}$, respectively. The integral in Eq. (1) represents the superposition of all the plane waves in the MS whose wave vectors lie on a cone of angle $\tan\theta_0 = k_t/k_z$. Observe that the wave vectors delineate the circumference of radius k_t on the MS; see Fig. 1. It is important to remark that Eq. (1) is the solution of the Helmholtz equation in any cylindrical coordinate system by performance of the corresponding coordinate transformation.

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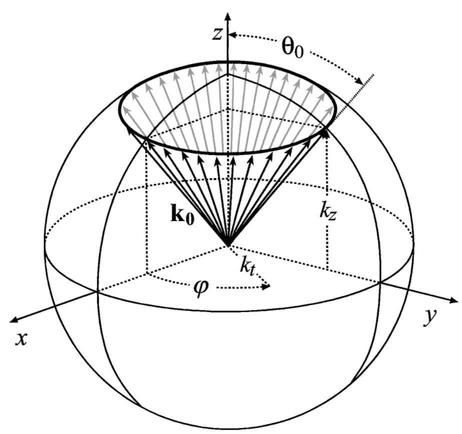

Fig. 1. IOF's are characterized by the ring formed by the intersection between the cone $\theta = \theta_0$ and the McCutchen sphere $|\mathbf{k}| = k_0$.

Because of the form of Eq. (1), it is clear that it represents the propagation of invariant transverse patterns.[4,5] Function $A(\varphi)$ defines the structure of the transverse pattern.[7] The case in which $A(\varphi)$ is constant was analyzed in Ref. 1, and the IOF is described by function $J_0(k_t r)$. When $A(\varphi)$ is of the form $\exp(in\varphi)$, the field can be written in terms of the higher Bessel functions $J_n(k_t r)\exp(in\varphi)$, and a linear superposition of solutions of this kind is also invariant.[7–9] When $A(\varphi)$ is a random function it produces IOF's with an irregular transverse intensity distribution.[10] Note that $A(\varphi) = \sum_{n=0}^{N-1} \delta(\varphi - 2\pi n/N)$, which produces kaleidoscopic patterns, corresponds to the case of the superposition of a discrete number of N plane waves regularly distributed on the circumference.

Now, to find the description of IOF's in elliptical cylindrical coordinates requires that we solve the Helmholtz equation. Elliptical cylindrical coordinates are defined by $x = h \cosh \xi \cos \eta$, $y = h \sinh \xi \sin \eta$, and $z = z$, where $\xi \in [0, \infty)$ and $\eta \in [0, 2\pi)$ are the radial and the angular variables, respectively, and $2h$ is the interfocal separation.[11,12] In these conditions the Helmholtz equation separates into a longitudinal part, which has a solution with dependence $\exp(ik_z z)$, and a transverse part, whose solution $u_t(\xi, \eta)$ must satisfy

$$\frac{\partial^2 u_t(\xi,\eta)}{\partial \xi^2} + \frac{\partial^2 u_t(\xi,\eta)}{\partial \eta^2} + \frac{h^2 {k_t}^2}{2}(\cosh 2\xi - \cos 2\eta)u_t(\xi,\eta) = 0\,, \quad (2)$$

which can be split into the radial and angular Mathieu differential equations.[6,11] k_t satisfies the dispersion relation ${k_0}^2 = {k_t}^2 + {k_z}^2$. From Eq. (2), the zero-order fundamental traveling-wave solutions are

$$u^{(1)}(\xi,\eta,z;q) = [Ce_0(\xi;q) + iFey_0(\xi;q)]ce_0(\eta;q)\exp(ik_z z)$$

$$u^{(2)}(\xi,\eta,z;q) = [Ce_0(\xi;q) - iFey_0(\xi;q)]ce_0(\eta;q)\exp(ik_z z)\,, \quad (3)$$

where Ce_0 and Fey_0 are the even radial Mathieu functions of the first and second kinds, respectively, ce_0 is the angular Mathieu function, and $q = h^2 {k_t}^2/4$ is a parameter related to the ellipticity of the coordinate system. The expressions within the brackets are the first and second Mathieu–Hankel functions of zero order.[6,11] We remark that higher-order mode solutions of Eq. (2) also exist with rotating phase features.[11–13] Equations (3) represent traveling conical waves, modulated azimuthally, as will be shown below, slanting outward, $u^{(1)}$, and inward, $u^{(2)}$, whose wave vectors form an angle $\theta_0 = \tan^{-1}(k_t/k_z)$ with respect to the z axis.

In infinite space, where both traveling conical wave solutions coexist and overlap, we have

$$u(\xi,\eta,z;q) = Ce_0(\xi;q)ce_0(\eta;q)\exp(ik_z z)\,, \quad (4)$$

which is an IOF that will be referred to as the zero-order Mathieu beam. Comparing Eq. (1) with this last result [Eq. (4)], we can deduce that for cylindrical elliptical coordinates the integral must be proportional to the product of the Mathieu functions in Eq. (4). In fact, when we set $A(\varphi) = ce_0(\varphi; q)$ within the integral of Eq. (1), transform (x, y) to polar coordinates (r, ϕ) by use of the relation $x \cos \varphi + y \sin \varphi = r \cos(\varphi - \phi)$, and use the Euler formula for the complex exponential, it is possible to evaluate the integral,[11–13] and the resulting field is proportional to $Ce_0(\xi; q)ce_0(\eta; q)$. This implies that if the cone of wave vectors in Eq. (1) is modulated by $ce_0(\varphi; q)$ the resulting IOF will be a Mathieu beam.

Until now we have assumed that the fields are of infinite extent; however, in real physical situations the fields are of finite transverse extension, introducing diffraction effects into the evolution of the otherwise invariant optical field. We have simulated the propagation of a truncated Mathieu beam, using as the initial condition Eq. (4) at $z = 0$, through a circular aperture. As we know of no available numerical libraries that one can use to compute the whole family of Mathieu functions, in particular the radial ones, we developed a library to compute them based on the theory of Mathieu functions.[11–13] After applying the coordinate transformations defined above, we used an algorithm to solve the Helmholtz equation.[14] We have chosen the parameters of the simulation to give a geometric maximum propagation distance of ~15.5 m. An aperture of radius 10 mm gives an angle $\theta_0 \approx 0.00065$ rad. Assuming that illumination at $\lambda = 632.8$ nm produces $k_t = k_0 \sin \theta_0 = 6324.64\ \mathrm{m}^{-1}$. We also set the interfocal distance of the elliptical coordinates such that $h = 1$ mm.

In Fig. 2(a) we show the transverse pattern of the truncated zero-order Mathieu beam for the given parameters, and in Fig. (b) its corresponding angular spectrum is shown. The evolution of the transverse intensity profiles along the planes x–z and y–z is shown in Fig. 3. One may observe the very peculiar evolution of the intensity in the x–z plane [Figs. 3(a) and 3(c)]; the beam propagates as a one-dimensional quasi-invariant beam. The sudden vanishing of the intensity is due to the fact that the superposition of

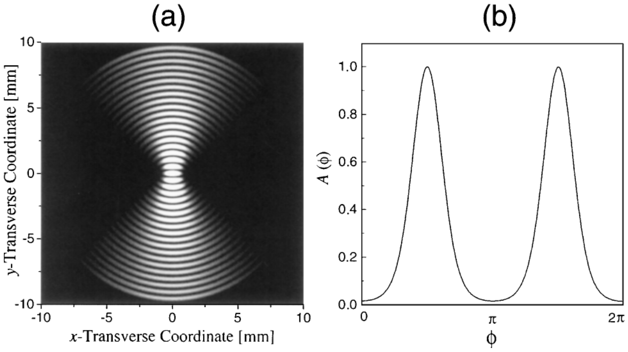

Fig. 2. (a) Transverse intensity pattern of a truncated zero-order Mathieu beam. (b) Angular spectrum of the zero-order Mathieu beam.

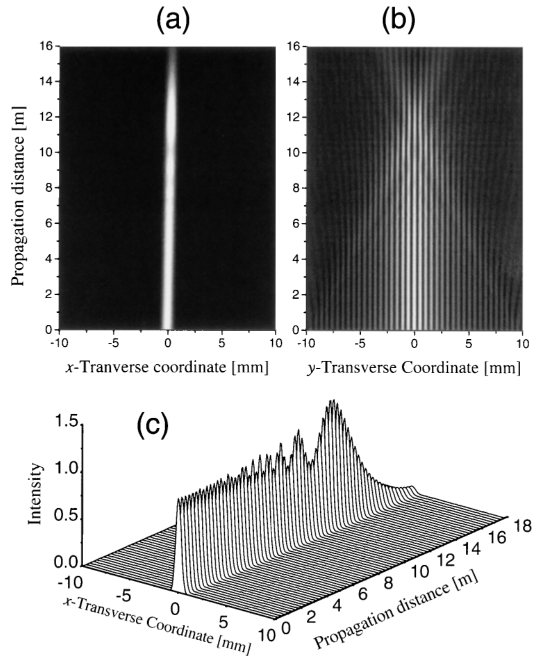

Fig. 3. Evolution of an apertured Mathieu beam on planes (a) $x-z$ and (b) $y-z$. (c) Another view of the evolution in plane $x-z$. One may observe the very well-defined quasi-invariant beam within the conic overlapping region.

conical waves that form the finite Mathieu beam can occur only within a finite conical region. The characteristic cone-shaped region is clearly observed in the perpendicular plane as shown in Fig. 3(b). The very well-defined pattern in one direction contrasts with the resulting pattern in the other direction, which has a transverse oscillating behavior, and the central peak does not differ much from the lateral structure. This is contrary to what is found for the J_0–Bessel beam, in which the central peak is much more intense than the surrounding structure. The effects of diffraction can clearly be observed in the behavior of the axial peak intensity and the variations along the edges of the region of invariance.

To create Mathieu beams in the laboratory we refer to the experimental setup suggested by Fig. 2(b), an annular slit with transmittance modulated by $A(\varphi) = ce_0(\varphi; q)$ and a lens, as in the experiment of Durnin *et al.*[1] This kind of transmittance can be difficult to achieve. However, the required transmittance function can be approximated by an annular slit of radius r_0 illuminated with a one-dimensional strip pattern with a Gaussian profile produced, for instance, with a cylindrical lens. Mathematically this function is represented by $\delta(r - r_0)\exp(-x^2/w^2)$, where width w is related to parameter q and can be adjusted according to the desired pattern. This feature is of great importance since it provides a very simple way to create a quasi-Mathieu beam in the laboratory.

In conclusion, we have shown that the Mathieu–Hankel functions, which form a whole set of exact traveling-wave solutions, can be used to describe a class of IOF's, Mathieu beams. Using the McCutchen theorem, we demonstrated the relation between the general class of IOF's and these beams. Analogously to plane waves and Bessel beams, Mathieu beams also form an orthogonal and complete set, in the sense that any invariant field can be represented as the superposition of Mathieu beams. Finally, we proposed a simple experimental setup for creating an approximation to this kind of beam.

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References

1. J. E. Durnin, J. J. Miceli, and J. H. Eberly, Phys. Rev. Lett. **58,** 1499 (1987).
2. E. G. Kalnins and W. Miller, Jr., J. Math. Phys. **17,** 331 (1976).
3. C. W. McCutchen, J. Opt. Soc. Am. **54,** 240 (1964).
4. G. Indebetouw, J. Opt. Soc. Am. A **6,** 150 (1989).
5. Y. Y. Ananev, Opt. Spectrosc. (USSR) **64,** 722 (1988).
6. J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).
7. A. Vasara, J. Turunen, and A. T. Friberg, J. Opt. Soc. Am. A **6,** 1748 (1989).
8. R. Piestun and J. Shamir, J. Opt. Soc. Am. A **15,** 3039 (1998).
9. Z. Bouchal and J. Wagner, Opt. Commun. **176,** 299 (2000).
10. J. Turunen, A. Vasara, and A. T. Friberg, J. Opt. Soc. Am. A **8,** 282 (1991).
11. N. W. McLachlan, *Theory and Applications of Mathieu Functions* (Oxford University, London, 1951).
12. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, 5th ed. (Academic, London, 1994).
13. J. C. Gutiérrez-Vega, "Formal analysis of the propagation of invariant optical fields with elliptical symmetries," Ph.D. dissertation (Instituto Nacional de Astrofísica, Optica y Electroníca, Puebla, Mexico, 2000); jgutierr@campus.mty.itesm.mx.
14. M. D. Feit and J. A. Fleck, Jr., J. Opt. Soc. Am. B **5,** 633 (1988).